

A CANONICAL BUNDLE FORMULA FOR PROJECTIVE LAGRANGIAN FIBRATIONS

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ABSTRACT. We classify singular fibres of a projective Lagrangian fibration over codimension one points. As an application, we obtain a canonical bundle formula for a projective Lagrangian fibration over a smooth manifold.

1. INTRODUCTION

We begin with the definitions of *symplectic varieties* and *Lagrangian fibrations*.

DEFINITION 1.1. *A normal Kähler variety X is said to be a symplectic variety if X satisfies the following two conditions:*

- (1) *There exists a nondegenerate holomorphic closed 2-form ω on the smooth locus U on X .*
- (2) *For any resolution $\nu : Y \rightarrow X$ of X , the pull back of ω to $\nu^{-1}(U)$ can be extended as a holomorphic 2-form on Y .*

DEFINITION 1.2. *Let X be a symplectic variety, ω a symplectic form on X and S a normal variety. A proper surjective morphism with connected fibres $f : (X, \omega) \rightarrow S$ is said to be a Lagrangian fibration if a general fibre F of f is a Lagrangian variety with respect to ω , that is, $\dim F = (1/2) \dim X$ and the restriction of the symplectic 2-form $\omega|_{F \cap X_{\text{smooth}}}$ is identically zero, where X_{smooth} is the smooth locus of X .*

If X is a smooth surface, a Lagrangian fibration is nothing but a minimal elliptic fibration. It is expected that these two objects share many geometric properties. From the view point of Minimal Model Program, it is natural to consider a Lagrangian fibration from a symplectic variety with only \mathbb{Q} -factorial terminal singularities. In [8, Theorem 2], singular fibres of a minimal elliptic fibration are classified. As a higher dimensional analogy, we obtain the following result.

THEOREM 1.3. *Let (X, ω) be a symplectic variety with only \mathbb{Q} -factorial terminal singularities. Assume that X admits a projective Lagrangian fibration $f : (X, \omega) \rightarrow \Delta^n$ over an n -dimensional polydisk Δ^n with the smooth discriminant locus D . Then*

- (1) *X is smooth.*
- (2) *The morphism $X \times_{\Delta^n} D \rightarrow D$ is decomposed $X \times_{\Delta^n} D \rightarrow A$ and $A \rightarrow D$, which is a smooth abelian fibration of relative dimension $n - 1$ except the following two cases.*

Type I_1 *The base change $X \times_{\Delta^n} D$ is reduced and irreducible. The normalization of $X \times_{\Delta^n} D$ is a \mathbb{P}^1 -bundle over a smooth abelian fibration $A \rightarrow D$. The original $X \times_{\Delta^n} D$ is obtained by patching two disjoint sections of the \mathbb{P}^1 -bundle.*

Type I_m *The base change $X \times_{\Delta^n} D$ is reduced and normal crossing. Each irreducible component is a \mathbb{P}^1 -bundle over a smooth abelian fibration $A \rightarrow D$. The dual graph is*

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Type	Irreducible components and their multiplicities	Fibre	Degree
$I_0-l, l = 2, 3, 4, 6$	lE_0	I_0	l
I_0^*-0	$2E_0 + E_1 + E_2 + E_3 + E_4$	I_0^*	1
I_0^*-2	$2E_0 + E_1 + E_2$	I_0^*	2
I_0^*-4	$2E_0 + E_1$	I_0^*	4
I_0^*-1	$4E_0 + 2E_1 + 2E_2 + 2E_3$	I_0^*	2
I_0^*-5	$4E_0 + 2E_1 + 2E_2$	I_0^*	4
I_0^*-3	$6E_0 + 3E_1 + 3E_2$	I_0^*	3
$I_m^*-0, (m \geq 1)$	$2 \sum_{i=1}^m + E_{m+1} + E_{m+2} + E_{m+3} + E_{m+4}$	I_m^*	1
$I_m^*-1, (m \geq 1)$	$2 \sum_{i=1}^m + E_{m+1} + E_{m+2}$	I_m^*	2
$I_m^*-2, (m \geq 1)$	$2 \sum_{i=1}^m E_i$	I_{2m}	4
$I_m^*-3, (m \geq 1)$	$2 \sum_{i=1}^m E_i$	I_{2m}	4
II	E_0	II	1
II^*	$6E_0 + 3E_1 + 4E_2 + 2E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8$	II^*	1
$III-0$	$E_0 + E_1$	III	1
$III-1$	E_0	III	2
III^*-0	$4E_0 + 3E_1 + 3E_2 + 2E_3 + 2E_4 + E_5 + E_6 + 2E_7$	III^*	1
III^*-1	$4E_0 + 3E_1 + 2E_2 + E_3 + 2E_4$	III^*	2
$IV-0$	$E_0 + E_1 + E_2$	IV	1
$IV-1$	E_0	IV	3
$IV-2$	$2E_0 + 2E_1$	IV	2
IV^*-0	$3E_0 + 2E_1 + 2E_2 + 2E_3 + E_4 + E_5 + E_6$	IV^*	1
IV^*-1	$3E_0 + 2E_1 + E_2$	IV^*	3
IV^*-2	$6E_0 + 4E_1 + 4E_2 + 2E_3 + 2E_4$	IV^*	2

TABLE 1. Classification of singular fibres I, The types listed in shaded rows does not admit local sections.

the extended Dynkin diagram of type \tilde{A}_m ($m \geq 2$). Each intersection of irreducible components forms a section of the \mathbb{P}^1 -bundle structure of both components.

If we have a decomposition $X \times_{\Delta}^n D \rightarrow A \rightarrow D$, every fibre of $X \times_{\Delta}^n D \rightarrow A$ is isomorphic to a singular fibre of a minimal elliptic surface over a curve. Moreover there exists a smooth abelian fibration \tilde{A} and a finite étale morphism $\tilde{A} \rightarrow A$ over D such that $(X \times_{\Delta}^n D) \times_A \tilde{A}$ is isomorphic to the product of a fibre of $X \times_{\Delta}^n D \rightarrow A$ and \tilde{A} . According to fibres of $X \times_{\Delta}^n D \rightarrow D$, the number of irreducible components, their multiplicities and the degree of $\tilde{A} \rightarrow A$, they are classified 23 types in Table 1.

(3) There exists examples of all types.

REMARK 1.4. If $\dim X = 4$, each type in Table 1 corresponds to the same named type of [10, Table 4 and 5] except I_0^*-5 , $I_m^*-0, 1, 2$ or 3. In [10], type I_0^*-5 is identifies as type I_0^*-2 , because the author does not consider multiplicities in [10]. The types $I_m^*-0, 1$ or 2 correspond to [10, Theorem 1.2 (2)] whose dual graph is of the Dynkin diagram of type A_m . In the proof of [10, Theorem 1.2], the author misses type I_m^*-3 because the argument of [10, (5.3.1) (2)] is incomplete. If $X \times_{\Delta}^n D$ is of type I_m^*-2 or 3, $(X \times_{\Delta}^n D)_{\text{red}}$ is a normal crossing variety whose dual graph is the Dynkin diagram of type A_m . Each edge component is non normal and is obtained patching

Type	$I_m, (m \geq 1)$	I_0-l	$I_0^*-1,2,3$	$I_0^*-4,5$	I_0^*-6	$I_m^*-1,2,3,4 (m \geq 1)$	II
a_P	0	$1 - 1/l$	$1/2$	$3/4$	$5/6$	$1/2$	$1/6$
Type	II^*	$III-1,2$	$III^*-1,2$	$IV-1,2$	$IV^*-1,2$	$IV-3$	IV^*-3
a_P	$5/6$	$1/4$	$3/4$	$1/3$	$2/3$	$2/3$	$5/6$

TABLE 2. The coefficients of the canonical bundle formula.

two \mathbb{P}^1 -bundles over a smooth abelian fibration if $X \times_{\Delta^n} D$ is of type I_m^*-3 , while it is smooth if $X \times_{\Delta^n} D$ is of type I_m^*-2 . An example is constructed in the proof of Theorem 1.3.

REMARK 1.5. Theorem 1.3 (1) does not mean that a symplectic variety with only \mathbb{Q} -factorial singularities is always smooth. This implies the singular locus of a Lagrangian fibration is concentrated in fibres over codimension two points. Namely, for a projective Lagrangian fibration $f : X \rightarrow S$ from a symplectic variety X with only \mathbb{Q} -factorial terminal singularities, let S° be the intersection of the smooth locus of S and the smooth locus of the discriminant locus of f . Then the preimage of S° is smooth.

REMARK 1.6. In the proof of Theorem 1.3, projectivity of a morphism is crucial. On the other hand, in [7, Theorem 1.3 and 1.4], they obtained a classification of singular fibres at a general point of the discriminant locus of non projective Lagrangian fibrations from smooth symplectic manifolds. They also give the information of leaves of foliations arising a symplectic form.

Using this classification, we obtain a concrete form of a canonical bundle formula of a projective Lagrangian fibration due to [4, Theorem 4.5].

COROLLARY 1.7. Let X be a projective symplectic variety with only \mathbb{Q} -factorial terminal singularities, S a projective smooth manifold and $f : (X, \omega) \rightarrow S$ a Lagrangian fibration. Then

$$K_X \sim f^*(K_S + L_{X/S}^{\text{ss}}) + \sum a_P f^* P$$

where P runs all irreducible components of the discriminant locus of f . The coefficients a_P are given in Table 2. Moreover $12L_{X/S}^{\text{ss}}$ is Cartier.

This paper is constructed as follows: In section 2, we prove that a projective Lagrangian fibration is bimeromorphic to a smooth or a projective abelian fibration of first order degenerations, which is defined in Definition 2.2, after finite base change. In sections 3 and 4, we construct a relative minimal model of the quotient of this projective abelian fibration, which is bimeromorphic to the original Lagrangian fibration. The proofs of Theorem 1.3 and Corollary 1.7 are given in section 5.

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2. SEMISTABLE REDUCTION

2.1. We start with the definition of *first order degenerations*, which is a mild degeneration of abelian varieties.

DEFINITION 2.2. A projective abelian fibration $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}^n$ over an n -dimensional polydisk $\tilde{\Delta}^n$ is said to be first order degenerations if \tilde{g} satisfies the following properties:

- (1) \tilde{Z} is smooth.
- (2) The discriminant locus \tilde{D} of \tilde{g} is smooth.
- (3) The base change $\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{D}$ is reduced and normal crossing. Moreover
 - (a) Each irreducible component is a \mathbb{P}^1 -bundle over a smooth abelian fibration $\tilde{A} \rightarrow \tilde{D}$.
 - (b) The dual graph is the extended Dynkin diagram of type \tilde{A}_m .
 - (c) Each intersection of irreducible components forms a section of the \mathbb{P}^1 -bundle structure of both components.
 - (d) Let \tilde{Z}_t be a fibre at $t \in \tilde{D}$ and \tilde{A}_t an intersection of irreducible components of \tilde{Z}_t . The automorphism of \tilde{A}_t defined by making a circuit of irreducible components along the ruling is a translation.

2.3. Next we investigate the monodromy matrix of a Lagrangian fibration and its action on the limit Hodge structure of a general fibre.

LEMMA 2.4. Let X and $f : X \rightarrow \Delta^n$ be as in Theorem 1.3. For a general point u of D , we choose a unit disk $\Delta \subset \Delta^n$ such that Δ intersects D at u transversally. Let \mathbb{H} be the upper half plane and $\mathbb{H} \rightarrow \Delta$ the map defined by $\tau \mapsto \exp(2\pi i\tau)$. We denote by X_∞ the base change $(X \times_{\Delta^n} \Delta) \times_\Delta \mathbb{H}$. Then

- (1) The limit Hodge structure of $H^1(X_\infty, \mathbb{C})$ is pure or

$$\dim H^{1,1} = \dim H^{0,0} = 1, \quad \dim H^{1,0} = \dim H^{0,1} = n - 1.$$

- (2) Let U be the monodromy matrix of $R^1 f_* \mathbb{C}|_{\Delta^n \setminus D}$ around D . The semisimple part U_s of U defines an automorphism of the mixed Hodge structure on $H^1(X_\infty, \mathbb{C})$. Moreover,

$$\dim(H^{0,1})^{U_s} \geq n - 1.$$

- (3) If U^m is unipotent, $\text{rank}(U^m - I) \leq 1$.

Proof. (1). We recall that a general fibre of f is an abelian variety by [9, Lemma 2.2]. Let $\nu : Y \rightarrow X$ be a resolution such that ν^*D is a simple normal crossing divisor and ν is isomorphic on $X \setminus f^{-1}(D)$. Since we choose u generally, the restriction $Y \times_{\Delta^n} \Delta \rightarrow X \times_{\Delta^n} \Delta$ is a resolution such that the fibre Y_u at u is a simple normal crossing divisor. Then we have a natural morphism

$$\alpha : H^1(Y \times_{\Delta^n} \Delta, \mathbb{C}) \rightarrow H^1(X_\infty, \mathbb{C}).$$

By [3, Theorem 6.9], Y_u is a deformation retract of $Y \times_{\Delta^n} \Delta$. Thus $H^1(Y \times_{\Delta^n} \Delta, \mathbb{C})$ carries the mixed Hodge structure which induced by $H^1(Y_u, \mathbb{C})$.

CLAIM 2.5. The morphism α is injective and compatible with the mixed Hodge structures.

Proof. By [6, Theorem 11*], if we choose a suitable degree of a cyclic cover $\tilde{\Delta} \rightarrow \Delta$ branching at u , then there exists a resolution \tilde{Y}_Δ of the normalization of $(Y \times_{\Delta^n} \Delta) \times_\Delta \tilde{\Delta}$ which is semistable. We have the following diagram:

$$\begin{array}{ccc} H^1(Y \times_{\Delta^n} \Delta, \mathbb{C}) & \xrightarrow{\alpha} & H^1(X_\infty, \mathbb{C}) \\ \beta \downarrow & & \parallel \\ H^1(\tilde{Y}_\Delta, \mathbb{C}) & \xrightarrow{\gamma} & H^1(X_\infty, \mathbb{C}) \end{array}$$

Since the normalization of $(Y \times_{\Delta^n} \Delta) \times_{\Delta} \tilde{\Delta}$ has only quotient singularities, β is injective. The morphism γ is a part of the Clemens-Schmid exact sequence. Thus γ is injective and compatible with the mixed Hodge structures. This implies α is injective and compatible with the mixed Hodge structures. \square

We go back to the proof of Lemma. By [9, Lemma 4.2 and 4.6],

$$\dim H^{1,0} = \dim F^1 \mathrm{Gr}_1^W H^1(Y_u, \mathbb{C}) \geq n - 1. \quad (1)$$

This implies the assertion (1) of Lemma.

(2) Since the monodromy of $R^1(\nu \circ f)_* \mathbb{C}|_{Y \times_{\Delta^n} \Delta \setminus Y_u}$ around u is U , U_s acts on $H^1(X_\infty, \mathbb{C})$ by [16, (2.13) Theorem]. Moreover the image of α is invariant under the action of U . By Claim 2.5 and the inequality (1), $H^{1,0}$ carries a U_s -invariant space whose dimension is greater than $n - 1$. The action of U_s is complex linear, we obtain the assertion (2) of Lemma.

(3) Since $(U^m - I)^2 = 0$, $U^m - I$ defines a morphism

$$W_2 H^1(X_\infty, \mathbb{C}) \rightarrow W_0 H^1(X_\infty, \mathbb{C})$$

by [16, (2.13) Theorem]. By the assertion (1) of this lemma, $\dim W_2 = 2n$ and $\dim W_0 = 0$ or 1 . This implies that $\mathrm{rank}(U^m - I) \leq 1$. \square

2.6. We prove that a projective Lagrangian fibration is bimeromorphic to a smooth or an abelian fibration of first order degenerations after a finite base change.

PROPOSITION 2.7. *Let $f : X \rightarrow \Delta^n$ be as in Lemma 2.4 and $\nu : Y \rightarrow X$ a resolution such that $(\nu \circ f)^* D$ is normal crossing. Then there exists a finite cyclic cover $\tilde{\pi} : \tilde{\Delta}^n \rightarrow \Delta^n$ which blanching along D and an abelian fibration $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}^n$ which has the following properties:*

- (1) *The morphism \tilde{g} is smooth or of first order degenerations.*
- (2) *It satisfies the following diagram:*

$$\begin{array}{ccc} (Y \times_{\Delta^n} \tilde{\Delta}^n)^\sim \xrightarrow{\mu} & & \tilde{Z} \\ \downarrow & & \downarrow \tilde{g} \\ \tilde{\Delta}^n & \xlongequal{\quad} & \tilde{\Delta}^n \end{array}$$

where $(Y \times_{\Delta^n} \tilde{\Delta}^n)^\sim$ is the normalization of $Y \times_{\Delta^n} \tilde{\Delta}^n$ and μ is bimeromorphic.

- (3) *Let H be the Galois group of π . The action of H on \tilde{Z} which induced by μ is holomorphic. Moreover \tilde{g} is H -equivariant.*

Proof. By [17, Proposition 6.1], if we choose a suitable order cyclic cover $\pi_1 : \tilde{\Delta}_1^n \rightarrow \Delta^n$ blanching along D , there exists a birational morphism $\nu_1 : \tilde{Y}_1 \rightarrow (Y \times_{\Delta^n} \tilde{\Delta}^n)^\sim$ such that the composition morphism $\tilde{f}_1 : \tilde{Y}_1 \rightarrow \tilde{\Delta}_1^n$ is semistable in codimension one. Let $\tilde{D}_1 = \pi_1^{-1}(D)$. By [9, Lemma 2.2], a general fibre of f is an abelian variety. Thus we have a multivalued map

$$p : \tilde{\Delta}_1^n \setminus \tilde{D}_1 \rightarrow \mathbb{H}_n,$$

where \mathbb{H}_n is the Siegel upper half space of degree n . We consider separately in cases that p can be extended holomorphically on $\tilde{\Delta}_1^n$ or not.

CLAIM 2.8. *If p is extended on $\tilde{\Delta}_1^n$ holomorphically, there exists a projective smooth abelian fibration $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}_1^n$. Moreover it satisfies (2) and (3) of the assertions of Proposition 2.7 if we put $\pi = \pi_1$ and $\tilde{\Delta}^n = \tilde{\Delta}_1^n$.*

Proof. By [13, Theorem (3.5)], we obtain a smooth projective abelian fibration $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}_1^n$ which is an extension of $\tilde{Y}_1 \setminus \tilde{f}_1^{-1}(\tilde{D}) \rightarrow \tilde{\Delta}_1^n \setminus \tilde{D}_1$. Then \tilde{Y}_1 and \tilde{Z} are bimeromorphic to each other over $\tilde{\Delta}_1^n$ by [11, Proposition 1.6]. Thus the Galois group H of π_1 acts on \tilde{Z} bimeromorphically and \tilde{g} is H -equivariant. Since every fibre of \tilde{g} has no rational curves, H acts on \tilde{Z} holomorphically. \square

We go back to the proof of Proposition. If p is not extended on $\tilde{\Delta}_1^n$, we need the following three Claims.

CLAIM 2.9. *If p is not extended on $\tilde{\Delta}_1^n$, then*

- (1) *There exists a semi abelian scheme $\tilde{f}' : G \rightarrow \tilde{\Delta}_1^n$ which is an extension of $\tilde{Y}_1 \setminus \tilde{f}_1^{-1}(\tilde{D}) \rightarrow \tilde{\Delta}_1^n \setminus \tilde{D}_1$.*
- (2) *For every fibre over \tilde{D}_1 , the rank of torus part is one.*

Proof. (1) We will construct the required family according to [13, Theorem (5.3)]. By [13, Definition (1.14)], the pair $(\tilde{\Delta}_1^n, \tilde{D}_1)$ is toroidal embedding. We have

$$M_\alpha = \mathbb{Z}, N_\alpha = \mathbb{Z}, \sigma_\alpha = \{y \geq 0; y \in N_{\alpha, \mathbb{R}} = N_\alpha \otimes_{\mathbb{Z}} \mathbb{R}\}$$

as the conical invariants associated to $(\tilde{\Delta}_1^n, \tilde{D}_1)$ in [13, (1.16)]. Since $\tilde{\Delta}_1^n$ and \tilde{D}_1 are smooth, the condition (R) in [13, Remark (4.4)] is satisfied. Since \tilde{f}_1 is projective, the condition (SG) in [13, Remark (4.4)] is satisfied. Since \tilde{f}_1 is semistable in codimension one, the monodromy of the local system $\bigcup_{s \in \tilde{\Delta}_1^n} H_1(\tilde{Y}_{1,s}, \mathbb{Z})$ is unipotent. Therefore the assumption (U) (ii) in [13, (4.3)] is satisfied and hence the all assumptions of (U) are satisfied by [13, Remark (4.4) (i), (ii)]. Thus, for every $t \in \tilde{D}_1$, $\lim_{t' \in \tilde{\Delta}_1^n \setminus \tilde{D}_1 \rightarrow t} p(t')$ lies in a rational boundary component F_α by [13, (4.4)]. According to [13, (2.7) and Definition (4.5)], we denote by \mathfrak{R}_α'' the vector space associated to F_α . If we give a cone decomposition of $N_{\alpha, \mathbb{R}} \times \mathfrak{R}_\alpha''$, we obtain an extended family. Let

$$\Sigma_\alpha := \{(u, v) \in N_{\alpha, \mathbb{R}} \times \mathfrak{R}_\alpha''; u > 0\}$$

Then this cone itself defines a cone decomposition of $N_{\alpha, \mathbb{R}} \times \mathfrak{R}_\alpha''$ over σ_α ([13, (1.19) Definition]). Now we apply [13, Theorem (5.3)] and obtain an abelian fibration $\tilde{f}' : G \rightarrow \tilde{\Delta}_1^n$. Then G is a semi-abelian scheme by [13, Theorem 5.3 4) f) and 5)].

(2) For the proof of the assertion, we set up notation. The Siegel upper half plane \mathbb{H}_n is realized as a bounded symmetric domain $D_n = \{C \in \mathrm{GL}(n, \mathbb{C}); E_n - C\bar{C} > 0\}$ via the Cayley transformation

$$\mathbb{H}_n \ni \tau \mapsto (\tau - \sqrt{-1}E_n)(\tau + \sqrt{-1}E_n)^{-1} \in D_n.$$

Let

$$F_{n'} := \left\{ \begin{pmatrix} C' & 0 \\ 0 & 1 \end{pmatrix}; C' \in D_{n'} \right\}.$$

Now we consider the rank of torus part. By [12, Definition (4.15)], for a rational boundary component F_α , there exists the integer n' and a matrix $M \in \mathrm{Sp}(n, \mathbb{Z})$ such that $MF_\alpha = F_{n'}$. By [13, Example (2.8) and Definition (4.5)], the rank of torus part equals to $n - n'$. Let $\tau(t) = Mp(t)$. Then

$$\lim_{t' \in \tilde{\Delta}_1^n \setminus \tilde{D}_1 \rightarrow t} \tau(t') \in F_{n'},$$

for every point t of \tilde{D}_1 by [13, (4.4)]. To investigate this limit, we consider the monodromy of U' of the local system $\bigcup_{t' \in \tilde{\Delta}_1^n \setminus \tilde{D}_1} H_1(\tilde{Y}_{1,t'}, \mathbb{Z})$ because this is also the monodromy of p . The matrix U' coincides with ${}^tU^{-m}$, where U is the monodromy of $R^1 f_* \mathbb{C}|_{\Delta^n \setminus D}$ and m is the degree

of $\pi_1 : \tilde{\Delta}_1^n \rightarrow \Delta^n$. Since \tilde{f}_1 is semistable, U^m is unipotent. This implies $\text{rank}(U' - I) = 1$ by Lemma 2.4 (3). By [12, Lemma (4.7)], $M^{-1}U'M \in \mathfrak{U}(F_{n'})$ and we have

$$M^{-1}U'M = \begin{pmatrix} E_{n-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & E_{n-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, b \in \mathbb{Q}$$

by [12, Proposition (4.8)]. Since the monodromy of $\tau(t)$ is $M^{-1}U'M$, we have

$$\tau((\exp 2\pi i)t') = \begin{pmatrix} \tau_{11}(t') & \tau_{12}(t') \\ {}^t\tau_{12}(t') & \tau_{22}(t') + b \end{pmatrix},$$

where $\exp(2\pi i)t'$ represents the point $t' \in \tilde{\Delta}_1^n \setminus \tilde{D}_1$ after moving a circuit around \tilde{D}_1 and each size of the matrixes τ_{11} , τ_{12} and τ_{22} is $(n-1) \times (n-1)$, $(n-1) \times 1$ and 1×1 , respectively. This implies that the matrix functions τ_{11}, τ_{12} are extended holomorphically on $\tilde{\Delta}_1^n$ and

$$\tau_{11}(t) = \lim_{t' \in \tilde{\Delta}_1^n \setminus \tilde{D}_1 \rightarrow t} \tau_{11}(t') \in \mathbb{H}_{n-1}.$$

By a direct calculation,

$$\lim_{t' \in \tilde{\Delta}_1^n \setminus \tilde{D}_1 \rightarrow t} (\tau(t') - \sqrt{-1}E_n)(\tau(t') + \sqrt{-1}E_n)^{-1} = \begin{pmatrix} C' & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$C' = (\tau_{11}(t) - \sqrt{-1}E_{n-1})(\tau_{11}(t) + \sqrt{-1}E_{n-1}).$$

This is an element of D_{n-1} and we obtain that $n' = n - 1$. □

CLAIM 2.10. *There exists a finite cyclic cover $\pi_2 : \tilde{\Delta}^n \rightarrow \tilde{\Delta}_1^n$ which branches along \tilde{D}_1 such that the base change $G \times_{\tilde{\Delta}_1^n} \tilde{\Delta}^n$ has a projective compactification $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}^n$ which is of first order degenerations.*

Proof. We will construct \tilde{Z} according to [15]. By [14, Corollary XI 1.16], $\tilde{f}' : G \rightarrow \tilde{\Delta}_1^n$ satisfies the conditions of [15, Définition 1]. By [2, page 31], we have the following exact sequence

$$0 \rightarrow T \rightarrow \tilde{G} \rightarrow A \rightarrow 0,$$

where T is a torus, \tilde{G} is a semi abelian scheme and A is an abelian scheme. By [15, Lemma 3 and Théorème 5], there exists a finite cyclic cover $\pi_2 : \tilde{\Delta}^n \rightarrow \tilde{\Delta}_1^n$ branching along \tilde{D}_1 which has the following two properties:

- (1) The base change $T \times_{\tilde{\Delta}_1^n} \tilde{\Delta}^n$ is a split torus.
- (2) There exists a projective abelian fibration $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}^n$, which is a compactification of $\tilde{G} \times_{\tilde{\Delta}_1^n} \tilde{\Delta}^n$.

We denote by $\tilde{\tilde{G}}$, $\tilde{\tilde{A}}$ and $\tilde{\tilde{T}}$ by the base changed objects and $\tilde{D} = \pi_2^{-1}(\tilde{D}_1)$. To prove that \tilde{g} is of first order degenerations, we trace a part of arguments in [15, §2.4 and 2.5], which consists of three steps.

Step 1.: Taking a relative compactification Z' of $\tilde{\tilde{T}}$.

Step 2.: Taking the contracted product $\tilde{\tilde{G}} \times^{\tilde{\tilde{T}}} Z'$, which is the quotient of $\tilde{\tilde{G}} \times_{\tilde{\tilde{A}}} Z'$ by the free action of $\tilde{\tilde{T}}$ with the standard action on the first factor and the opposite action on the second factor.

Step 3.: Taking the quotient of $\tilde{\tilde{G}} \times^{\tilde{\tilde{T}}} Z'$ by the action of \mathbb{Z}^r , where r is the rank of $\tilde{\tilde{T}}$.

First we consider Step 1. By Claim 2.9 (2), the rank of \tilde{T} is one. Thus it is same to consider a relative complete model of \tilde{T} and a relative complete model of degenerations of elliptic curves. Therefore every $t \in \tilde{D}$, the fibre Z'_t is a infinite chain of \mathbb{P}^1 . Moreover Z'_t is reduced. Next we consider Step 2. Let \tilde{g} be the morphism $\tilde{G} \times^{\tilde{T}} Z' \rightarrow \tilde{\Delta}^n$. Then the divisor $\tilde{g}^* \tilde{D}$ is reduced and each irreducible component is a \mathbb{P}^1 -bundle over \tilde{A} . Finally we consider Step 3. By Claim 2.9 (2), we consider the quotient by \mathbb{Z} . Let t be a point of \tilde{D} . The action of \mathbb{Z} on the fibre Z'_t at t is cycling through the chain and it on the fibre \tilde{A}_t at t of $\tilde{A} \rightarrow \tilde{\Delta}^n$ is a translation. Therefore $\tilde{Z} = \tilde{G} \times^{\tilde{T}} Z' / \mathbb{Z} \rightarrow \tilde{\Delta}^n$ satisfies the all conditions of first order degenerations. \square

CLAIM 2.11. *The morphism \tilde{g} satisfies the assertions (2) and (3) of Proposition 2.7 if we put $\tilde{\pi} = \pi_1 \circ \pi_2$.*

Proof. We consider the base change $Y \times_{\Delta^n} \tilde{\Delta}^n$. By [11, Propsition 1.6], the normalization of $Y \times_{\Delta^n} \tilde{\Delta}^n$ and \tilde{Z} are bimeromorphic to each other and satisfy the diagram in Proposition 2.7. Thus H acts on \tilde{Z} bimeromorphically. Since the action of H preserves fibres of \tilde{g} and every fibre of \tilde{g} contains no flopping curves, the action of H is holomorphic. \square

We complete the proof of Proposition 2.7. \square

PROPOSITION 2.12. *We use same notations as in Lemma 2.4 and Proposition 2.7. Let t be a point of $\pi^{-1}(D)$, \tilde{Z}_t the fibre at t of \tilde{g} , and $H^{0,1}$ the piece of the mixed Hodge structure of $H^1(\tilde{Z}_t, \mathbb{C})$. Then*

$$\dim(H^{0,1})^H \geq n - 1.$$

Proof. It is enough to prove that the statement for a general point $t \in \pi^{-1}(D)$. We set up notations. For a general point t , we choose unit disks $\Delta \subset \Delta^n$ and $\tilde{\Delta} \subset \tilde{\Delta}^n$ which satisfies the following two properties:

$$(1) \pi(\tilde{\Delta}) = \Delta$$

$$(2) \Delta \cap D = s \text{ and } \tilde{\Delta} \cap \pi^{-1}(D) = t, \text{ where } s = \pi(t). \text{ The intersections are normal crossing.}$$

Let \mathbb{H} be the upper half plain. For a morphism $\mathbb{H} \rightarrow \tilde{\Delta}$ which is defined by $\tau \mapsto \exp(2\pi i \tau)$, let $X_\infty = (X \times_{\Delta^n} \Delta) \times_{\Delta} \mathbb{H}$ and $\tilde{Z}_\infty = (\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{\Delta}) \times_{\tilde{\Delta}} \mathbb{H}$. We investigate the action of H on those spaces. Since $\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{\Delta}$ is a compactification of $(X \times_{\Delta^n} (\Delta - \{s\})) \times_{(\Delta - \{s\})} (\tilde{\Delta} - \{t\})$, there is an H -equivariant isomorphism $X_\infty \cong \tilde{Z}_\infty$. Thus the isomorphism of the mixed Hodge structures

$$H^1(\tilde{Z}_\infty, \mathbb{C}) \rightarrow H^1(X_\infty, \mathbb{C})$$

is H -equivariant. On the other hand, there is a natural morphism $k : \tilde{Z}_\infty \rightarrow \tilde{Z} \times_{\tilde{\Delta}^n} \tilde{\Delta}$. Since $\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{\Delta} \rightarrow \tilde{\Delta}$ is semistable, k induces the first term of the Clemens-Schmid exact sequence

$$0 \rightarrow H^1(\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{\Delta}, \mathbb{C}) \rightarrow H^1(\tilde{Z}_\infty, \mathbb{C}).$$

Since k is H -equivariant and the mixed Hodge structure of $H^1(\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{\Delta}, \mathbb{C})$ is induced by that of $H^1(\tilde{Z}_t, \mathbb{C})$, we obtain a H -equivariant injection of the mixed Hodge structures

$$0 \rightarrow H^1(\tilde{Z}_t, \mathbb{C}) \rightarrow H^1(X_\infty, \mathbb{C}).$$

We note that the above morphism is isomorphic if we take Gr_1^W . By [16, Theorem 2.3], the action on H on the limit Hodge structure of $H^1(X_\infty, \mathbb{C})$ coincides with the action of U_s . Therefore we obtain the assertion of Proposition by Lemma 2.4. \square

3. SMOOTH CASES

3.1. Let $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}^n$ and H be as in Proposition 2.7. In this section, we will construct the relative minimal model of the quotient \tilde{Z}/H if \tilde{g} is smooth.

PROPOSITION 3.2. *Let $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}^n$, D , $\pi : \tilde{\Delta}^n \rightarrow \Delta^n$ and H be as in Proposition 2.7. Assume that \tilde{g} is smooth.*

- (1) *The quotient \tilde{Z}/H has the unique minimal model $g : Z \rightarrow \Delta^n$. Moreover Z is smooth.*
- (2) *If K_Z is trivial, $Z \times_{\Delta^n} D$ has the properties of one of types in Table 1 other than I_m^*-0 , 1, 2 and 3.*

Proof. First we consider the quotient of \tilde{Z} by a subgroup H' of H , which is defined by the following Lemma.

LEMMA 3.3. *There exists the subgroup H' of H such that H' acts on $H^1(\tilde{Z}_t, \mathcal{O})$ trivially and H/H' acts on $H^1(\tilde{Z}_t, \mathcal{O})$ faithfully for every $t \in \pi^{-1}(D)$, where \tilde{Z}_t is the fibre at t .*

Proof. Since \tilde{g} is H -equivariant and H acts on $\pi^{-1}(D)$ trivially, H acts on \tilde{Z}_t for every $t \in \pi^{-1}(D)$. Let h be a generator of H and $U(t)$ the representation matrix of the action of h on $H^1(\tilde{Z}_t, \mathcal{O})$. We denote by $a_i(t_0)$, $(1 \leq i \leq n)$ eigenvalues of $U(t)$. The matrix functions U_t and $a_i(t)$ are holomorphic with respect to t . Since H is a finite group, there exists a positive minimal integer m such that $U(t)^m = I$ for every $t \in \pi^{-1}(D)$. Thus $U(t)$ is diagonalizable and every $a_i(t)$ satisfies the equation $a_i^m(t) = 1$. This implies $a_i(t)$ are constants and $U(t)^{m'} \neq I$ for every $0 < m' < m$ and every t . Therefore if we define H' the subgroup of H generated by h^m , H' satisfies the assertions of Lemma. \square

LEMMA 3.4. *Let H' be as in Lemma 3.4. We denote by \bar{Z} and $\bar{\Delta}^n$ the quotients \tilde{Z} and $\tilde{\Delta}^n$ by H' , respectively.*

- (1) *The quotient \bar{Z} is smooth. The reduced structure of every fibre of $\bar{g} : \bar{Z} \rightarrow \bar{\Delta}^n$ is an abelian variety.*
- (2) *Let \bar{D} be the discriminant locus of \bar{g} . Then there exists an integer l such that $\bar{g}^* \bar{D} = l(\bar{g}^* \bar{D})_{\text{red}}$ and*

$$K_{\bar{Z}} \sim (l-1)(\bar{g}^* \bar{D})_{\text{red}}.$$

- (3) *If $H \neq H'$, for every point s of \bar{D} ,*

$$\dim H^1((\bar{Z}_s)_{\text{red}}, \mathcal{O})^{H/H'} = n-1,$$

where \bar{Z}_s is the fibre at s .

Proof. (1) Let h be an element of H' . By Lemma 3.3 (1), the action of h on \tilde{Z}_t is a translation. Thus the fixed locus is $\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{D}$ or empty. This implies \bar{Z} is smooth and the reduced structure of every fibre is an abelian variety.

(2) Since every fibre is irreducible, $\bar{g}^* \bar{D} = l(\bar{g}^* \bar{D})_{\text{red}}$ for a suitable integer l . By the adjunction formula,

$$K_{\bar{Z}} + (\bar{g}^* \bar{D})_{\text{red}}|_{(\bar{g}^* \bar{D})_{\text{red}}} \sim K_{(\bar{g}^* \bar{D})_{\text{red}}}$$

On the other hand, $K_{(\bar{g}^* \bar{D})_{\text{red}}}$ is trivial, because $(\bar{g}^* \bar{D})_{\text{red}} \rightarrow \bar{D}$ is a smooth abelian fibration. Thus we obtain the assertion.

(3) Let $t = \pi^{-1}(s)$. By Proposition 2.12 and the assumption of Lemma, we have

$$\dim H^1(\tilde{Z}_t, \mathcal{O})^H = n-1.$$

Since the quotient morphism $\tilde{Z}_t \rightarrow (\bar{Z}_s)_{\text{red}}$ is étale, we have

$$H^1((\bar{Z}_s)_{\text{red}}, \mathcal{O})^{H/H'} = H^1(\tilde{Z}_t/H', \mathcal{O})^{H/H'} \cong H^1(\tilde{Z}_t, \mathcal{O})^H,$$

and we are done. \square

3.5. We will consider the quotient $\bar{Z}/(H/H')$ instead by \tilde{Z}/H . Let $\bar{Z}_{\bar{D}} = \bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ and $\bar{H} = H/H'$. Then $(\bar{Z}_{\bar{D}})_{\text{red}}$ admits a smooth abelian fibration $(\bar{Z}_{\bar{D}})_{\text{red}} \rightarrow \bar{D}$. Before considering the quotient \bar{Z}/\bar{H} , we investigate the action of \bar{H} on this fibration.

LEMMA 3.6. *Let $(\bar{Z}_{\bar{D}})_{\text{red}}$ and \bar{H} be as in 3.5. Assume that \bar{H} is not trivial. Then they satisfies the following diagram:*

$$\begin{array}{ccccc} (\bar{Z}_{\bar{D}})_{\text{red}} & \xrightarrow{\alpha} & \bar{A} & \xrightarrow{\beta} & \bar{D} \\ \downarrow & & \downarrow & & \downarrow \\ (\bar{Z}_{\bar{D}})_{\text{red}}/\bar{H} & \xrightarrow{\quad} & A & \xrightarrow{\gamma} & \bar{D}/\bar{H} \end{array} ,$$

where

- (1) β and γ are smooth morphisms whose fibres are $(n-1)$ -dimensional abelian varieties.
- (2) α is \bar{H} -equivariant.
- (3) $A \cong \bar{A}/\bar{H}$.

Proof. Let $(\bar{Z}_s)_{\text{red}}$ be the fibre of $\bar{g} : \bar{Z} \rightarrow \bar{\Delta}^n$ at $s \in \bar{D}$ with reduced structure and $(\bar{Z}_s)_{\text{red}}/\bar{H}$ is its quotient. We will prove

$$\dim H^1((\bar{Z}_s)_{\text{red}}/\bar{H}, \mathcal{O}) = n-1. \quad (2)$$

Since $(\bar{Z}_s)_{\text{red}}/\bar{H}$ is a V-manifold, it carries the coherent sheaf $\bar{\Omega}^1$ such that

$$\dim H^0((\bar{Z}_s)_{\text{red}}/\bar{H}, \bar{\Omega}^1) = \dim H^1((\bar{Z}_s)_{\text{red}}/\bar{H}, \mathcal{O})$$

and

$$H^0((\bar{Z}_s)_{\text{red}}/\bar{H}, \bar{\Omega}^1) = H^0((\bar{Z}_s)_{\text{red}}, \Omega^1)^{\bar{H}}$$

by [16, (1.6) and (1.8)]. Thus the equation (2) follows from Lemma 3.4 (3). Then we consider the relative Albanese map

$$(\bar{Z}_{\bar{D}})_{\text{red}}/\bar{H} \rightarrow A$$

over \bar{D}/\bar{H} . If we take the Stein factorization of $(\bar{Z}_{\bar{D}})_{\text{red}} \rightarrow A$, we obtain the diagram which satisfies all assertions of Lemma. \square

3.7. Let \bar{Z}_s and \bar{A}_s be the fibres at $s \in \bar{D}$. By the equation (2), $H^1(\bar{Z}_s, \mathbb{C})$ carries the $(2n-2)$ -dimensional sub pure Hodge structure, which is invariant under the action of \bar{H} . By Lemma 3.3, the action of \bar{H} on $H^1(\bar{Z}_s, \mathbb{C})$ is faithful. Since $\dim H^1(\bar{Z}_s, \mathbb{C}) = 2n$, the order of \bar{H} is 2, 3, 4 or 6. We define $\bar{H}'(s)$ be the kernel of the representation $\bar{H} \rightarrow \text{Aut} \bar{A}_s$. Since \bar{H} is a finite group and the action of H on A_s is a translation by Lemma 3.6, $\bar{H}'(s)$ does not depend on s and we denote simply \bar{H}' by this group. Now we consider the quotient of \bar{Z}/\bar{H} separately in cases of

- (1) \bar{H}' is trivial.
- (2) $\bar{H}' = \bar{H}$.
- (3) \bar{H}' is not trivial and $\bar{H}' \neq \bar{H}$.

The cases (1), (2) and (3) are treated in Lemma 3.8, 3.9 and 3.12 respectively.

LEMMA 3.8. *Assume that \bar{H}' is trivial. Then the quotient \bar{Z}/\bar{H} itself is the unique relative minimal model Z over Δ^n and smooth. If K_Z is trivial, then $Z \times_{\Delta^n} D$ satisfies all properties of Type I_0 - l , $l = 2, 3, 4, 6$ in Table 1.*

Proof. By the assumption, every element of \bar{H} induces a non trivial translation on \bar{A} . Thus the action of \bar{H} has no fixed points and the quotients \bar{Z}/\bar{H} is a relative minimal model over Δ^n . This is the unique relative minimal model because every fibre contains no rational curves. Assume that K_Z is trivial. Since the quotient morphism $\eta : \bar{Z} \rightarrow \bar{Z}/\bar{H} = Z$ is étale, we have

$$\eta^* K_Z \sim K_{\bar{Z}} \sim \mathcal{O}.$$

By Lemma 3.4, $\bar{Z} \times_{\Delta^n} \bar{D}$ is reduced. On the other hand, the order of \bar{H} is 2, 3, 4 or 6 by 3.7. Therefore $Z \times_{\Delta^n} D$ satisfies the all properties of Type I_0 - l in Table 1. \square

LEMMA 3.9. *Assume that $\bar{H} = \bar{H}'$. Then the quotient \bar{Z}/\bar{H} has the unique relative minimal model Z over Δ^n . Moreover Z is smooth. If K_Z is trivial, $Z \times_{\Delta^n} D$ satisfies all properties of one of types $I_0^*-0, 2, 4$, II , II^* , III -1, 2, III^* -1, 2, IV -1, 2, IV^* -1 or 2 in Table 1.*

Proof. By the assumption, every fibre of $(\bar{Z}_{\bar{D}})_{\text{red}} \rightarrow \bar{A}$ is stable under the action of \bar{H} . By Lemma 3.3, the action of \bar{H} on each fibre is a multiplication. Thus every connected component of the fixed locus of $(\bar{Z}_{\bar{D}})_{\text{red}}$ forms a multisection of $(\bar{Z}_{\bar{D}})_{\text{red}} \rightarrow \bar{A}$. We note that if one of connected component of the fixed locus forms a section of $(\bar{Z}_{\bar{D}})_{\text{red}} \rightarrow \bar{A}$, then $(\bar{Z}_{\bar{D}})_{\text{red}}$ is isomorphic to the product of \bar{A} and an elliptic curve. The quotient $(\bar{Z}_{\bar{D}})_{\text{red}}/\bar{H} \rightarrow A$ is a \mathbb{P}^1 -bundle and each connected component of the singular locus of \bar{Z}/\bar{H} forms a multisection of the \mathbb{P}^1 -bundle. Each connected component of fixed locus is also forms a multisection of $(\bar{Z}_{\bar{D}})_{\text{red}} \rightarrow \bar{A}$. Each singular point is locally isomorphic to the product of a surface singularity F and \mathbb{C}^{2n-2} . Since a \mathbb{P}^1 -bundle with three sections is the product of \mathbb{P}^1 and a base space, there exists a smooth abelian fibration \tilde{A} and a finite étale morphism $\tilde{A} \rightarrow A$ such that $((\bar{Z}/\bar{H})_{\text{red}} \times_{\Delta^n} D) \times_A \tilde{A}$ is the product of \mathbb{P}^1 and \tilde{A} . According to these properties, the singularities of \bar{Z}/\bar{H} is classified 13 types in Table 3. The rest of the proof of Lemma is divided the following two Claims.

CLAIM 3.10. *If the singularities is of type $I_0^*-1, 2, 3$, II^* , $III^*-1, 2$, IV^*-1 or 2, there exists the unique relative minimal model Z of \bar{Z}/\bar{H} over Δ^n . Moreover Z is smooth. If K_Z is trivial, then $Z \times_{\Delta^n} D$ satisfies all properties of one of types in Table 1, which coincides with the type of the singularities.*

Proof. Using the minimal resolution of F , we obtain the minimal resolution $\nu : W \rightarrow \bar{Z}/\bar{H}$. Since F is isomorphic to a Du Val singularity, ν is crepant. Thus we put $Z = W$. This is the unique relative minimal model, because every fibre of $Z \rightarrow \Delta^n$ contains no flopping curve. By the construction, every fibre of $Z \times_{\Delta^n} D \rightarrow A$ is isomorphic to the Kodaira singular fibre of type I_0^* , II^* , III^* or IV^* . By Table 3, $Z \times_{\Delta^n} D$ satisfies the assertions of Claim if $\bar{Z} \times_{\Delta^n} \bar{D}$ is reduced. Thus we will prove it under the assumption that K_Z is trivial. Since ν is crepant, $K_Z \sim \nu^* K_{\bar{Z}/\bar{H}}$. Thus

$$K_{\bar{Z}/\bar{H}} \sim \nu_* \nu^* K_{\bar{Z}/\bar{H}} \sim \nu_* K_Z \sim \mathcal{O}.$$

Let $\eta : \bar{Z} \rightarrow \bar{Z}/\bar{H}$ be the quotient morphism. Since η is étale in codimension one, we have

$$K_{\bar{Z}} \sim \eta^* K_{\bar{Z}/\bar{H}} \sim \mathcal{O}.$$

This implies $\bar{Z} \times_{\Delta^n} \bar{D}$ is reduced by Lemma 3.4. \square

Type	The order of \bar{H}	Singular locus of \bar{Z}/\bar{H}	F	Degree of $\tilde{A} \rightarrow A$
I_0^*-0	2	Four sections	$(1/2)(1, 1)$	1
I_0^*-2	2	Two 2-sections	$(1/2)(1, 1)$	2
I_0^*-4	2	A 4-section	$(1/2)(1, 1)$	4
II	6	A section	$(1/6)(1, 1)$	1
		A section	$(1/3)(1, 1)$	
		A section	$(1/2)(1, 1)$	
II^*	6	A section	$(1/6)(1, 5)$	1
		A section	$(1/3)(1, 2)$	
		A section	$(1/2)(1, 1)$	
$III-1$	4	Two sections	$(1/4)(1, 1)$	1
		A section	$(1/2)(1, 1)$	
$III-2$	4	A 2-section	$(1/4)(1, 1)$	2
		A section	$(1/2)(1, 1)$	
III^*-1	4	Two sections	$(1/4)(1, 3)$	1
		A section	$(1/2)(1, 1)$	
III^*-2	4	A 2-section	$(1/4)(1, 3)$	2
		A section	$(1/2)(1, 1)$	
$IV-1$	3	Three sections	$(1/3)(1, 1)$	1
$IV-2$	3	A 3-section	$(1/3)(1, 1)$	3
IV^*-1	3	Three sections	$(1/3)(1, 2)$	1
IV^*-2	3	A 3-section	$(1/3)(1, 2)$	3

TABLE 3. Classification of singular locus I

CLAIM 3.11. *If the singularities is of type II , $III-1, 2$, $IV-1$ or 2 , there exists the unique relative minimal model Z of \bar{Z}/\bar{H} over Δ^n . Moreover Z is smooth. If K_Z is trivial, then $Z \times_{\Delta^n} D$ satisfies all properties of one of types in Table 1, which coincides with the types of the singularities.*

Proof. Using the minimal resolution of surface singularities, we obtain the minimal resolution $\nu : W \rightarrow \bar{Z}/\bar{H}$. Then we obtain the minimal model Z by the same process as in [1, page 209]. This is the unique relative minimal model, because every fibre of $Z \rightarrow \Delta^n$ contains no flopping curve. Moreover Z is smooth. By the construction, every fibre of $Z \times_{\Delta^n} D \rightarrow A$ is isomorphic to the Kodaira singular fibre of type II , III or IV . By Table 3, $Z \times_{\Delta^n} D$ satisfies the assertions of Claim if $\bar{Z} \times_{\Delta^n} \bar{D}$ is reduced. Thus we will prove it under the assumption that K_Z is trivial. By a direct calculation,

$$K_{\bar{Z}/\bar{H}} \sim \ell((\bar{g}/\bar{H})^* D)_{\text{red}}, \begin{cases} \ell = 4 & \text{Type } II, \\ \ell = 2 & \text{Type } III-1, 2, \\ \ell = 1 & \text{Type } IV-1, 2, \end{cases}$$

where $(\bar{g}/\bar{H}) : \bar{Z}/\bar{H} \rightarrow \Delta^n$ is the quotient of the morphism of \bar{g} . Let $\eta : \bar{Z} \rightarrow \bar{Z}/\bar{H}$ be the quotient morphism. Since η is étale in codimension one,

$$K_{\bar{Z}} \sim \ell(\bar{g}^* \bar{D})_{\text{red}}$$

If the multiplicity of $\bar{Z} \times_{\Delta^n} \bar{D}$ is l , then $\ell + 1 \equiv 0 \pmod{l}$ by Lemma 3.4. Thus $l = 1$ or 5 if the singularities of \bar{Z}/\bar{H} is of type II . We derive a contradiction assuming that $l = 5$. Let u

Type	The order of \bar{H}	The order of \bar{H}'	Singular locus of \bar{Z}/\bar{H}'	Type of singularities
I_0^*-1	4	2	Four sections	$(1/2)(1, 1)$
I_0^*-5	4	2	Two 2-sections	$(1/2)(1, 1)$
I_0^*-3	6	2	Four sections	$(1/2)(1, 1)$
I_0^*-7	4	2	A 4-section	$(1/2)(1, 1)$
I_0^*-8	6	2	Two 2-section	$(1/2)(1, 1)$
I_0^*-9	6	2	A 4-section	$(1/2)(1, 1)$
$IV-3$	6	3	Three sections	$(1/3)(1, 1)$
$IV-4$	6	3	A 3-section	$(1/3)(1, 1)$
IV^*-3	6	3	Three sections	$(1/3)(1, 2)$
IV^*-4	6	3	A 3-section	$(1/3)(1, 2)$

TABLE 4. Classification of singular locus II

be a point of D , Z_u the fibre at u and Δ a unit disk which intersects D at u transversally. We consider the morphism $g_\Delta : Z \times_{\Delta^n} \Delta \rightarrow \Delta$. Since K_Z is trivial, the canonical divisor $K_{Z \times_{\Delta^n} \Delta}$ is also trivial. On the other hand, $5(Z_u)_{\text{red}} = Z_u$ as a divisor of $Z \times_{\Delta^n} \Delta$. By the adjunction formula, $K_{(Z_u)_{\text{red}}}$ should be 5-torsion. However $(Z_u)_{\text{red}}$ is the product of the Kodaira singular fibre of type II and an abelian variety. Thus $K_{(Z_u)_{\text{red}}}$ is trivial. That is a contradiction. In the other cases, by the same argument, we obtain contradictions if we assume that $l \neq 1$. \square

We complete the proof of Lemma 3.9. \square

LEMMA 3.12. *Assume that \bar{H}' is not trivial and $\bar{H} \neq \bar{H}'$. Then the quotient \bar{Z}/\bar{H} has the unique relative minimal model Z over Δ^n . Moreover Z is smooth. If K_Z is trivial, $Z \times_{\Delta^n} D$ satisfies all properties of one of Type $I_0^*-1, 3, 5$, $IV-3$ or IV^*-3 in Table 1.*

Proof. By the same argument of the proof of Lemma 3.9, the singularities of \bar{Z}/\bar{H}' is classified as in Table 4.

CLAIM 3.13. *The singularities listed in the shaded rows of Table 4 does not occur.*

Proof. We derive a contradiction assuming these singularities actually occur. Let $(\bar{Z}/\bar{H})_u$ be the fibres at $u \in D$ and A_u the fibre of $A \rightarrow D$ at u . Then $(\bar{Z}/\bar{H})_u$ is a \mathbb{P}^1 -bundle over A_u and the singular locus of \bar{Z}/\bar{H} defines multisections of this \mathbb{P}^1 -bundle. There exists an abelian variety \tilde{A}_u and a finite étale morphism $\tilde{A}_u \rightarrow A_u$ such that $(\bar{Z}/\bar{H})_u \times_{A_u} \tilde{A}_u$ is isomorphic to the product of \mathbb{P}^1 and \tilde{A}_u . We note that the degree of $\tilde{A}_u \rightarrow A_u$ is greater than 6 by Table 4. Then we consider the pull back of the singular locus of \bar{Z}/\bar{H} to $(\bar{Z}/\bar{H})_u \times_{A_u} \tilde{A}_u$. These loci are preserved by the action of the Galois group J of $\tilde{A}_u \rightarrow A_u$. Since $(\bar{Z}/\bar{H})_u \times_{A_u} \tilde{A}_u \cong \mathbb{P}^1 \times \tilde{A}_u$, J defines an action on \mathbb{P}^1 which preserves three or four points on \mathbb{P}^1 . On the other hand, J is commutative and its order is greater than 6, such an action does not exist. That is a contradiction. \square

We go back to the proof of Lemma. Using the minimal resolution of surface singularities, we obtain the minimal resolution $\nu : W \rightarrow \bar{Z}/\bar{H}$. If the singularities is of type $I_0^*-1, 3, 5$ or IV^*-3 , ν is crepant and we put $Z = W$. If the singularities is of type $IV-3$, we obtain a relative minimal model Z by the same process in [1, page 209]. The obtained Z is smooth and the unique relative minimal model over Δ^n because every fibre contains no flopping curves. If we prove that $\bar{Z} \times_{\Delta^n} \bar{D}$

is reduced, the rest of the assertions follows. We will prove it by assuming that K_Z is trivial. If the singularities is of type $I_0^*-1, 3, 5$ or IV^*-3 , we obtain $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is reduced by the same argument in the proof of Claim 3.10. If the singularities is of type $IV-3$, the multiplicity of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is 1 or 2 by the same argument in the proof of Claim 3.11. We derive a contradiction assuming the multiplicity is 2. Let u be a point of D , Z_u the fibre at u and Δ a unit disk which intersect D at u transversally. We consider the morphism $g_\Delta : Z \times_{\Delta^n} \Delta \rightarrow \Delta$. Since K_Z is trivial, the canonical divisor $K_{Z \times_{\Delta^n} \Delta}$ is trivial. On the other hand, $4(Z_u)_{\text{red}} = Z_u$ as a divisor of $Z \times_{\Delta^n} \Delta$. By the adjunction formula, $K_{(Z_u)_{\text{red}}}$ should be 4-torsion. However there exists an étale cover of degree two of $(Z_u)_{\text{red}}$ which is the product of the Kodaira singular fibre of type IV and an abelian variety. Thus $K_{(Z_u)_{\text{red}}}$ is 2-torsion. That is a contradiction. \square

We complete the proof of Proposition 3.2. \square

4. FIRST ORDER DEGENERATION CASES

4.1. Let $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}^n$ and H be as in Proposition 2.7. In this section, we will construct the relative minimal model of the quotient \tilde{Z}/H if \tilde{g} is of first order degenerations. Before to investigate the quotient, we prepare three Lemmas.

LEMMA 4.2. *Let $\sum_{i=0}^l E_i$ be a pure dimensional normal crossing variety such that*

- (1) *The dual graph of $\sum E_i$ is the extended Dynkin diagram of type \tilde{A}_m .*
- (2) *Each irreducible component E_i is a \mathbb{P}^1 -bundle over an abelian variety.*
- (3) *Each intersection $E_i \cap E_j$ forms a section of a \mathbb{P}^1 -bundle.*
- (4) *The automorphism of each intersection of irreducible components defined by making a circuit of irreducible components along the ruling is a translation.*

Then

- (1) *The dualizing sheaf is locally free and trivial.*
- (2) *Let $j_i : E_i \rightarrow \sum E_i$ and $j_{ij} : E_i \cap E_j \rightarrow \sum E_i$ be the inclusions. Then j_i and j_{ij} induce the isomorphisms*

$$\text{Gr}_1^W H^1(\sum E_i, \mathbb{C}) \rightarrow H^1(E_i, \mathbb{C})$$

and

$$\text{Gr}_1^W H^1(\sum E_i, \mathbb{C}) \rightarrow H^1(E_i \cap E_j, \mathbb{C}).$$

- (3) *The condition (4) is equivalent to $\dim H^{0,1} = \dim(\sum E_i) - 1$, where $H^{0,1}$ is the piece of the mixed Hodge structure of $H^1(\sum E_i, \mathbb{C})$.*

Proof. (1) Since the dual graph is the extended Dynkin diagram of type \tilde{A}_m , we may assume that each irreducible component is indexed by each element of $\mathbb{Z}/m\mathbb{Z}$. Let $n : \coprod E_i \rightarrow \sum E_i$ be the normalization. Then we have

$$n^* \omega_{\sum E_i} \sim \prod \omega_{E_i}((E_{i-1} + E_{i+1})) \sim \mathcal{O},$$

by the properties (2) and (3) of $\sum E_i$. If there exists a section $(\prod s_i) \in \prod \Gamma(\omega_{E_i}((E_{i-1} + E_{i+1})))$ such that

$$\text{Res}_{E_i \cap E_{i+1}}(s_i) = \text{Res}_{E_i \cap E_{i+1}}(s_{i+1}), \quad (3)$$

then we are done. We denote by A the intersection $E_0 \cap E_{m-1}$. By tracing the ruling of E_i with decreasing order, we define a morphism $p_i : E_i \rightarrow A$ for $0 \leq i \leq m-1$. Then

$$\omega_{E_i}(E_{i-1} + E_{i+1}) \cong p_i^* \omega_A \otimes \omega_{E_i/A}(E_{i-1} + E_{i+1}).$$

Hence if we give sections of sheaves of the right hand side, we obtain a section of $\omega_{E_i}(E_{i-1} + E_{i+1})$. Since $\omega_{E_i/A}(E_{i-1} + E_{i+1})$ is trivial, there is a section $s'_i \in \Gamma(E_i, \omega_{E_i/A}(E_{i-1} + E_{i+1}))$. Let ω_0 be a section of ω_A and we define

$$s_i = p_i^* \omega_0 \otimes s'_i.$$

Then $p_{m-1}^* \omega_0|_A = \omega_0$ by the property (4) of $\sum E_i$. Thus we obtain the sections s_i which satisfies the equation (3) if we choose suitable s'_i .

(2) Let $j'_{ii+1} : E_i \rightarrow E_i \cap E_{i+1}$ the inclusion morphism. By [5, Ch 4.],

$$\mathrm{Gr}_1^W H^1(\sum E_i, \mathbb{C}) = \mathrm{Ker} \left(\bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} H^1(E_i, \mathbb{C}) \xrightarrow{\delta} \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} H^1(E_i \cap E_{i+1}, \mathbb{C}) \right), \quad (4)$$

where δ is defined by

$$\delta = \oplus (-1)^i j'_{ii+1}{}^*.$$

By the properties (2) and (3) of $\sum E_i$, every $j'_{ii+1}{}^*$ is isomorphic. Thus we obtain the assertion (2) by the definition of δ and the property (4) of $\sum E_i$.

(3) By the assertion (2) of Lemma, $\dim H^{0,1} = \dim(\sum E_i) - 1$. On the other hand, if $\sum E_i$ satisfies the conditions (1), (2) and (3), but not (4), then the automorphism induced on each intersection by making a circuit of irreducible components along the ruling is not trivial on its cohomology. Thus $\dim H^{0,1} < \dim(\sum E_i) - 1$ by the equation (4). \square

LEMMA 4.3. *Let $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}^n$ be a projective abelian fibration over an n -dimensional polydisk $\tilde{\Delta}^n$ with the smooth discriminant \tilde{D} , which satisfies the properties of Definition 2.2 except $\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{D}$ is reduced. Assume that a finite cyclic group H acts on \tilde{Z} and $\tilde{\Delta}^n$ holomorphically. Moreover we assume that this action has the following properties:*

- (1) *The quotient $\tilde{\Delta}^n/H$ is isomorphic to a polydisk Δ^n . The quotient morphism $\tilde{\Delta}^n \rightarrow \Delta^n$ is a cyclic cover ramified along \tilde{D} .*
- (2) *The morphism \tilde{g} is H -equivariant.*
- (3) *For every point t of \tilde{D} , the induced isomorphism on the piece $H^{0,1}$ of the mixed Hodge structure $H^1(\tilde{Z}_t, \mathbb{C})$ is trivial, where \tilde{Z}_t is the fibre at t .*

Let $\Gamma_{\tilde{Z}}$ be the dual graph of $\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{D}$ and H' the kernel of the representation $H \rightarrow \mathrm{Aut}(\Gamma_{\tilde{Z}})$. Then there exists a projective abelian fibration $\bar{g} : \bar{Z} \rightarrow \bar{\Delta}^n$ over an n -dimensional polydisk $\bar{\Delta}^n$ with the smooth discriminant locus \bar{D} which satisfies the following properties:

- (1) *The fibration \bar{g} satisfies the properties of Definition 2.2 except $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is reduced, but $K_{\bar{Z}}$ is nef.*
- (2) *The fibration \bar{g} satisfies the following diagram:*

$$\begin{array}{ccc} \tilde{Z}/H' & \xrightarrow{\mu} & \bar{Z} \\ \bar{g} \downarrow & & \downarrow \bar{g} \\ \tilde{\Delta}^n/H' & \xlongequal{\quad} & \bar{\Delta}^n, \end{array}$$

where μ is bimeromorphic.

- (3) *The action of H/H' on \bar{Z} induced by μ is holomorphic. Moreover $\bar{g} : \bar{Z} \rightarrow \bar{\Delta}^n$ satisfies the above properties (1), (2) and (3) if we replace $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}^n$ and H by $\bar{g} : \bar{Z} \rightarrow \bar{\Delta}^n$ and H/H' .*

Proof. First we investigate the singularities of \tilde{Z}/H' . Let \tilde{A} be an intersection of irreducible components of $\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{D}$ and ϕ the isomorphism on \tilde{Z} induced by the action of H' . By the assumption of the action of H' on \tilde{Z} , ϕ preserves \tilde{A} . By the assumption of \tilde{Z} , there exists a smooth abelian fibration $\tilde{A} \rightarrow \tilde{D}$. We note that every fibre of $\tilde{A} \rightarrow \tilde{D}$ is preserved by ϕ , because \tilde{g} is H -equivariant and H acts on \tilde{D} trivially. We will prove that ϕ is a translation on each fibre of $\tilde{A} \rightarrow \tilde{D}$. Let t be a point of \tilde{D} and \tilde{A}_t the fibre at t . By Lemma 4.2 and the assumption (3) of the action of H on \tilde{Z} , ϕ induces the identity map on $H^1(\tilde{A}_t, \mathcal{O})$. This implies that ϕ induces a translation on \tilde{A}_t . Since ϕ preserves every irreducible component of $\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{D}$ and its \mathbb{P}^1 -bundle structure, a connected component of fixed locus of ϕ coincides with an irreducible component or an intersection of them. Hence the quotient $\tilde{g}/H' : \tilde{Z}/H' \rightarrow \tilde{\Delta}^n/H'$ has singularities along intersections of irreducible components of $(\tilde{Z}/H') \times_{\tilde{\Delta}^n/H'} \tilde{D}/H'$. The singularity is locally isomorphic to the product of a surface quotient singularity and \mathbb{C}^{2n-2} .

Next we construct the minimal model of this quotient. We take the minimal resolution $\nu : W \rightarrow \tilde{Z}/H'$. Then H/H' acts on W holomorphically and the morphism $W \rightarrow \tilde{\Delta}^n/H'$ is H/H' -equivariant. We run a H/H' -equivariant relative minimal model program over $\tilde{\Delta}^n/H'$. Then each contraction morphism is a divisorial contraction, because all rational curves of varieties appeared in this process are fibres of \mathbb{P}^1 -bundles which form divisors. Thus we obtain the relative minimal model $\bar{g} : \bar{Z} \rightarrow \bar{\Delta}^n/H'$. It is obvious that \bar{g} satisfies the assertions (1) and (2) of Lemma. To prove that \bar{g} satisfies (3), we fix notation. Let \bar{D} be the discriminant locus of \bar{g} , s a point of \bar{D} , $\pi : \tilde{\Delta}^n \rightarrow \bar{\Delta}^n = \tilde{\Delta}^n/H'$ the quotient morphism and $t = \pi^{-1}(s)$. We choose a unit disk $\bar{\Delta} \subset \bar{\Delta}^n$ which intersects \bar{D} transversally at s . Then we consider the following diagram

$$\begin{array}{ccccccc}
\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{\Delta} & \xrightarrow{\alpha} & (\tilde{Z}/H') \times_{\tilde{\Delta}^n} \tilde{\Delta} & \xleftarrow{\beta} & W \times_{\tilde{\Delta}^n} \tilde{\Delta} & \xrightarrow{\gamma} & \bar{Z} \times_{\tilde{\Delta}^n} \bar{\Delta} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{\Delta} & \xrightarrow{\quad} & \tilde{\Delta} & \xlongequal{\quad} & \tilde{\Delta} & \xlongequal{\quad} & \tilde{\Delta}
\end{array}$$

By the Leray spectral sequence, we have an injection

$$0 \rightarrow H^1((\tilde{Z}/H')_s, \alpha_* \mathbb{C}) \rightarrow H^1(\tilde{Z}_t, \mathbb{C}).$$

Since $\alpha_* \mathbb{C}^{H'} \cong \mathbb{C}$, we have an injection

$$H^1((\tilde{Z}/H')_s, \mathbb{C}) \rightarrow H^1(\tilde{Z}_t, \mathbb{C}).$$

Since this morphism is compatible with the mixed Hodge structures, we have the action of H on the piece $H^{0,1}$ of the mixed Hodge structure of $H^1((\tilde{Z}/H')_s, \mathbb{C})$ is trivial. Then we consider the following diagram:

$$\begin{array}{ccccc}
H^1((\tilde{Z}/H')_s, \mathbb{C}) & \xrightarrow{\quad} & H^1(W_s, \mathbb{C}) & \xleftarrow{\quad} & H^1(\bar{Z}_s, \mathbb{C}) \\
\uparrow & & \uparrow \mu_1^* & & \uparrow \mu_2^* \\
H^1((\tilde{Z}/H') \times_{\tilde{\Delta}^n} \tilde{\Delta}, \mathbb{C}) & \xrightarrow{\beta^*} & H^1(W \times_{\tilde{\Delta}^n} \tilde{\Delta}, \mathbb{C}) & \xleftarrow{\gamma^*} & H^1(\bar{Z} \times_{\tilde{\Delta}^n} \bar{\Delta}, \mathbb{C}).
\end{array}$$

Since $R^1\beta_* \mathbb{C} = 0$ and $R^1\gamma_* \mathbb{C} = 0$, β^* and γ^* are isomorphic. Moreover, μ_1 and μ_2 are isomorphic because W_s and \bar{Z}_s are deformation retracts of $W \times_{\tilde{\Delta}^n} \tilde{\Delta}$ and $\bar{Z} \times_{\tilde{\Delta}^n} \bar{\Delta}$, respectively. Thus we have a surjection and an isomorphism

$$H^1((\tilde{Z}/H')_s, \mathbb{C}) \rightarrow H^1(W_s, \mathbb{C}) \cong H^1(\bar{Z}_s, \mathbb{C}).$$

Since these morphism is compatible with the mixed Hodge structures and H/H' -equivariant, we obtain the assertion (3) of Lemma. \square

LEMMA 4.4. *Let $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}^n$ be a projective abelian fibration which satisfies the properties of Definition 2.2 except $\tilde{Z} \times_{\tilde{\Delta}} \tilde{D}$ is reduced, but $K_{\tilde{Z}}$ is nef. Then*

- (1) $\tilde{g}^* \tilde{D} = l(\tilde{g}^* \tilde{D})_{\text{red}}$.
- (2) $K_{\tilde{Z}} \sim (l-1)(\tilde{g}^* \tilde{D})_{\text{red}}$.

Proof. (1) We start with fixing notation. Let t be a point of \tilde{D} and $\tilde{\Delta} \subset \tilde{\Delta}^n$ a unit disk which intersects \tilde{D} at t transversally. We denote by $\tilde{Z}_{\tilde{\Delta}}$ the base change $\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{\Delta}$ and by \tilde{Z}_t the fibre at t . As a divisor, let $\tilde{Z}_t = \sum e_i \tilde{E}_{i,t}$. It is enough to prove that every e_i equals to each other. Assume that e_0 is the maximum. By the adjunction formula,

$$K_{\tilde{E}_{0,t}} \equiv K_{\tilde{Z}_{\tilde{\Delta}}} - \frac{e_1}{e_0} \tilde{E}_{1,t}|_{\tilde{E}_{0,t}} - \frac{e_{m-1}}{e_0} \tilde{E}_{m-1,t}|_{\tilde{E}_{0,t}}$$

where $\tilde{E}_{1,t}$ and $\tilde{E}_{m-1,t}$ are the adjacent components. Since each $\tilde{E}_{i,t}$ is a \mathbb{P}^1 -bundle over an abelian variety and intersections form sections of \mathbb{P}^1 -bundle structures, we have

$$K_{\tilde{E}_{0,t}} \equiv -\tilde{E}_{1,t}|_{\tilde{E}_{0,t}} - \tilde{E}_{m-1,t}|_{\tilde{E}_{0,t}}.$$

Thus

$$K_{\tilde{Z}_{\tilde{\Delta}}}|_{\tilde{E}_0} \equiv -\left(1 - \frac{e_1}{e_0}\right)(\tilde{E}_{1,t}|_{\tilde{E}_{0,t}}) - \left(1 - \frac{e_{m-1}}{e_0}\right)(\tilde{E}_{m-1,t}|_{\tilde{E}_{0,t}})$$

If $e_{m-1} < e_0$ or $e_1 < e_0$, then $K_{\tilde{Z}_{\tilde{\Delta}}}.F < 0$ for a fibre F of the ruling of $\tilde{E}_{0,t}$. This is a contradiction because $K_{\tilde{Z}_{\tilde{\Delta}}} = K_{\tilde{Z}}|_{\tilde{Z}_{\tilde{\Delta}}}$ is nef. Thus $e_0 = e_1 = e_{m-1}$ and $K_{\tilde{Z}}|_{E_0} \equiv 0$. Repeating this argument, we obtain the assertion of Lemma.

(2) By the assertion of (1), we may assume that $\tilde{Z}_t = l \sum \tilde{E}_{i,t}$. It is enough to prove that

$$K_{\tilde{Z}_{\tilde{\Delta}}} \sim (l-1)(\tilde{Z}_t)_{\text{red}}.$$

By the above argument, $K_{\tilde{Z}_{\tilde{\Delta}}} \equiv 0$. Thus $K_{\tilde{Z}_{\tilde{\Delta}}} \sim \ell(\tilde{Z}_t)_{\text{red}}$ for an integer ℓ . Since $(\tilde{Z}_t)_{\text{red}}$ is a normal crossing variety which satisfies the all properties of Lemma 4.2, its dualizing sheaf is trivial. By the adjunction formula,

$$(\ell+1)(\tilde{Z}_t)_{\text{red}} \sim \mathcal{O}.$$

Therefore $\ell+1 = l$ and we are done. \square

4.5. Now we investigate the quotient \tilde{Z}/H , where \tilde{Z} is of first order degenerations.

PROPOSITION 4.6. *Let $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}^n$, D , $\pi : \tilde{\Delta}^n \rightarrow \Delta^n$ and H as in Proposition 2.7. Assume that \tilde{g} is of first degenerations. We denote by \tilde{D} the discriminant locus of \tilde{g} and by $\Gamma_{\tilde{Z}}$ the dual graph of $\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{D}$. If the action of H on $\Gamma_{\tilde{Z}}$ is rotation, we have the following.*

- (1) *The quotient \tilde{Z}/H is smooth. It is also the unique minimal model Z over Δ^n .*
- (2) *If K_Z is trivial, then $Z \times_{\Delta^n} D$ satisfies the properties of type I_m , ($m \geq 1$) in Theorem 1.3.*

Proof. (1) Let H' be the kernel of the representation $H \rightarrow \text{Aut}(\Gamma_{\tilde{Z}})$. By Lemma 4.3, there exists a projective fibration $\bar{g} : \bar{Z} \rightarrow \bar{\Delta}^n$ which is bimeromorphic to \tilde{Z}/H' . We consider the quotient $\bar{Z}/(H/H')$ instead of \tilde{Z}/H , because these are bimeromorphic to each other. If the representation of H/H' on the dual graph of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is not faithful, where \bar{D} is the discriminant locus of \bar{g} , we repeat the above process. Thus we may assume that the action of H/H' on the dual graph is faithful. Then the action has no fixed points, because this action on the dual graph

is a rotation by the assumption. Thus the quotient $\bar{Z}/H/H'$ is smooth and relatively minimal over Δ^n . This is the unique relative minimal model because every fibre contains no flopping curves.

(2) By the construction, it is enough to prove that $Z \times_{\Delta^n} D$ is reduced. Let u be a point of D and Z_u the fibre at u . If we prove that

$$\dim H^{0,1} = n - 1, \quad (5)$$

where $H^{0,1}$ is the piece of the mixed Hodge structure of $H^1(Z_u, \mathbb{C})$, then $g : Z \rightarrow \Delta^n$ satisfies the properties of Definition 2.2 except $Z \times_{\Delta^n} D$ is reduced by the construction and Lemma 4.2 (3). By Lemma 4.4,

$$K_Z \sim (l - 1)(Z \times_{\Delta^n} D)_{\text{red}},$$

where l is the multiplicity of $Z \times_{\Delta^n} D$. Since K_Z is trivial, $l = 1$ and this implies that $Z \times_{\Delta^n} D$ is reduced. Therefore we prove the equation (5). Let $s = \bar{\pi}^{-1}(u)$, where $\bar{\pi} : \bar{\Delta}^n \rightarrow \Delta^n$ is the quotient morphism. By Lemma 4.3, the action of H/H' on the piece of $H^{0,1}$ of the mixed Hodge structure of $H^1(\bar{Z}_s, \mathbb{C})$ is trivial. Since the quotient morphism $\bar{Z}_s \rightarrow Z_u$ is étale, $H^1(\bar{Z}_s, \mathbb{C})^{H/H'} \cong H^1(Z_u, \mathbb{C})$. Thus $\dim H^{0,1} = n - 1$. \square

PROPOSITION 4.7. *We use same notations as in Proposition 4.6. If the action of H on $\Gamma_{\bar{Z}}$ is reflection, we have the following.*

- (1) *The quotient \tilde{Z}/H has the unique minimal model $g : Z \rightarrow \Delta^n$. Moreover Z is smooth.*
- (2) *If K_Z is trivial, then $Z \times_{\Delta^n} D$ satisfies the all properties of one of types $I_m^*-0, 1, 2$ or 3 in Table 1.*

Proof. Let H' be the kernel of the representation $H \rightarrow \text{Aut}(\Gamma_{\bar{Z}})$. By the same argument in the proof of Proposition 4.6, we may assume that there exists a projective abelian fibration $\bar{g} : \bar{Z} \rightarrow \bar{\Delta}^n$ which satisfies the assertion of Lemma 4.3 and the action of H/H' on the dual graph of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is faithful, where \bar{D} is the discriminant locus of \bar{g} . We consider separately in cases that the action of H/H' has fixed points or no fixed points. They are treated in Lemma 4.8 and Lemma 4.10.

LEMMA 4.8. *If the action of H/H' on \bar{Z} has fixed points, the quotient $\bar{Z}/(H/H')$ has the unique relative minimal model Z on Δ^n . Moreover Z is smooth. If K_Z is trivial, $Z \times_{\Delta^n} D$ satisfies all properties of one of types I_m^*-0 or 1 in Table 1.*

Proof. We start with investigation of fixed locus.

CLAIM 4.9. *Under the assumptions of Lemma, we have the following.*

- (1) *The action preserves two irreducible components of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$. The number of irreducible components of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is even.*
- (2) *The morphism $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{D}$ is decomposed $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{A}$ and $\bar{A} \rightarrow \bar{D}$. Every fibre of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{A}$ is the Kodaira singular fibre of type I_{2m} and $\bar{A} \rightarrow \bar{D}$ is a smooth abelian fibration. Moreover these fibrations are H/H' -equivariant.*
- (3) *If $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is reduced, there exists a smooth abelian fibration $\tilde{A} \rightarrow \bar{D}$ and an étale morphism $\tilde{A} \rightarrow \bar{A}$ over \bar{D} of degree two such that $(\bar{Z} \times_{\bar{\Delta}^n} \bar{D}) \times_{\bar{A}} \tilde{A} \rightarrow \tilde{A}$ is isomorphic to the Kodaira singular fibre of type I_{2m} and \tilde{A} .*

Proof. (1) If we prove the first statement, the second follows from it. Thus we prove that H/H' preserves two irreducible components of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$. Since the action is a reflection, it preserves

intersections of irreducible components or irreducible components. We derive a contradiction assuming that an intersection is preserved. Let \bar{A} be the preserved intersection of irreducible components of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$. We prove that the action of H/H' on \bar{A} is trivial. Let s be a point of \bar{D} and \bar{A}_s the fibre at s . Then the natural inclusion $\bar{A}_s \rightarrow \bar{Z}_s$ is H/H' -equivariant. Thus the action of H/H' on $H^1(\bar{A}_s, \mathcal{O})$ is trivial by Lemma 4.3 and the assumption that the action of H/H' on the piece $H^{0,1}$ of the mixed Hodge structure of $H^1(\bar{Z}_s, \mathbb{C})$ is trivial. Since the action of H/H' has fixed points, the action of H/H' on \bar{A}_s should be trivial. This implies that the action of H/H' on \bar{A} is trivial. Therefore the dual graph of the singular fibre of $\bar{Z}/(H/H')$ is a chain and singularities of $\bar{Z}/(H/H')$ around $\bar{A}/(H/H')$ are locally isomorphic to the product the Du Val singularity of type A_1 and \mathbb{C}^{2n-2} . We take the minimal resolution $\nu : W \rightarrow \bar{Z}/(H/H')$ along $\bar{A}/(H/H')$. Then $W \times_{\Delta^n} D$ is a normal crossing variety whose dual graph is a chain. As a divisor,

$$W \times_{\Delta^n} D = E_0 + 2E_1 + 2E_2 + (\text{other components}),$$

where E_0 is the exceptional divisor such that $\nu(E_0) = A$, E_1 is the adjacent component of E_0 and E_2 is the adjacent component of E_1 . Each E_i is a \mathbb{P}^1 -bundle over $\bar{A}/(H/H')$ and intersects other components along sections. By Lemma 4.4, $K_{\bar{Z}}$ is numerically trivial. Thus $K_W|_{E_1} \equiv 0$. By the adjunction formula,

$$K_{E_1} \equiv -\frac{1}{2}E_0|_{E_1} - E_2|_{E_1}.$$

Since $E_0 \cap E_1$ and $E_2 \cap E_1$ are sections of the \mathbb{P}^1 -bundle structure of E_1 , this is a contradiction.

(2) By the assertion of (1), there exists two irreducible component which are stable under the action of H/H' . We denote by \bar{E}_1 and \bar{E}_2 these components. Let p be a point of \bar{E}_1 . There are two paths from p to \bar{E}_2 along the ruling of each irreducible component of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$. We denote by q_1 and q_2 the ends of the above two paths. By the same argument in the proof of (1), the action of H/H' on the $H^1(\bar{E}_i, \mathcal{O})$, ($i = 1, 2$) is trivial. Thus every fibre of the ruling of \bar{E}_2 is stable under the action of H/H' . Assume q_1 and q_2 does not lie on the same fibre. Since q_1 and q_2 are mapped each other by the action of H/H' , this is a contradiction. Hence there exists a fibration $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{A}$. The rests of the assertions are obvious.

(3) First we prove that every irreducible component of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is isomorphic to each other. By the assumption, there exists a section o of $\bar{g} : \bar{Z} \rightarrow \bar{\Delta}$ and we fix o . Let $\bar{Z}' = \bar{Z} \setminus \bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ and \bar{Z}° the largest open set such that the restriction morphism $\bar{g}|_{\bar{Z}^\circ} : \bar{Z}^\circ \rightarrow \bar{\Delta}$ is smooth. Then \bar{Z}' is an abelian scheme and \bar{Z}° is the Neron model of \bar{Z}' . Thus there exists a multiplicative morphism

$$\bar{Z}^\circ \times_{\bar{\Delta}^n} \bar{Z}^\circ \rightarrow \bar{Z}^\circ.$$

By this multiplicative morphism, we define a bimeromorphic map on \bar{Z} over $\bar{\Delta}^n$ for each section of $\bar{Z}^\circ \rightarrow \bar{\Delta}^n$ by [11, Proposition 1.6]. Since \bar{Z} is the unique relative minimal model over $\bar{\Delta}^n$, it defines an automorphism on \bar{Z} . Since the action of \bar{Z}° on \bar{Z}° is transitive, every component of \bar{Z}_s is mapped to each other. This implies every component of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is isomorphic to each other. Next we prove that there exists a smooth abelian fibration $\tilde{A} \rightarrow \bar{A}$ which satisfies the assertion of Claim. Let \bar{E}_1 and \bar{E}_2 be preserved components by the action of H/H' . Every fibre of the ruling of \bar{E}_1 and \bar{E}_2 is stable under the action of H/H' by the argument in the proof of the assertion (2) of Claim. Moreover the order of H/H' is two. Thus a connected component of the fixed locus of those components forms a section or a 2-section. If it forms a section, \bar{E}_i , ($i = 1, 2$) is isomorphic to the product of \mathbb{P}^1 and \bar{A} , because each \mathbb{P}^1 -bundle has at least three sections. If it forms 2-section, there exists a smooth projective abelian fibration $\tilde{A} \rightarrow \bar{D}$ and

an étale morphism $\tilde{A} \rightarrow \bar{A}$ over \bar{D} of degree two such that $\bar{E}_i \times_{\bar{A}} \tilde{A}$, ($i = 1, 2$) is isomorphic to the product of \mathbb{P}^1 and \bar{A} , because each base change has at least three sections. Since we have already proved that each irreducible component of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is isomorphic to each other, we are done. \square

We go back to the proof of Lemma. By Claim 4.9 (1) and (2), we have a fibration $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{A}$ and the induced automorphism on \bar{A} is trivial. Hence $(\bar{Z}/(H/H')) \times_{\Delta^n} D$ is a normal crossing variety and each connected component of singular locus of $\bar{Z}/(H/H')$ forms a multisection of edge components of $(\bar{Z}/(H/H')) \times_{\Delta^n} D$. The singularities of $\bar{Z}/(H/H')$ are locally isomorphic to the product of the Du Val singularity of type A_1 and \mathbb{C}^{2n-2} . We take the minimal resolution $\nu : Z \rightarrow \bar{Z}/(H/H')$. Then Z is the unique relative minimal model because Z has no flopping curves. Moreover $Z \times_{\Delta^n} D$ satisfies the properties of one of types I_m^* -0 or 1 in Table 1 except multiplicities. If we prove that $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is reduced, $Z \times_{\Delta^n} D$ satisfies all properties of one of types I_m^* -0 or 1. We prove it under the assumption that K_Z is trivial. Since ν is crepant, $K_{\bar{Z}/(H/H')}$ is trivial. The quotient morphism $\eta : \bar{Z} \rightarrow \bar{Z}/(H/H')$ is étale in codimension one. Thus

$$\eta^* K_{\bar{Z}/(H/H')} \sim K_{\bar{Z}}.$$

and we have that $K_{\bar{Z}}$ should be trivial. This implies that $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is reduced by Lemma 4.3. \square

LEMMA 4.10. *If the action of H/H' on \bar{Z} has no fixed points, the quotient $\bar{Z}/(H/H')$ is smooth. It is also the unique relative minimal model Z on Δ^n . In addition, $Z \times_{\Delta^n} D$ satisfies all properties of one of types I_m^* -2 or 3 in Table 1 if K_Z is trivial.*

Proof. By the assumption, the quotient $\bar{Z}/(H/H')$ is smooth and relatively minimal over $\bar{\Delta}^n/(H/H') = \Delta^n$. This is the unique relative minimal model because every fibre contains no flopping curves. We put $Z = \bar{Z}/(H/H')$. For the proof of the rest of assertions, we need the following two Claims.

CLAIM 4.11. *If K_Z is trivial, then $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is reduced.*

Proof. Since the quotient $\eta : \bar{Z} \rightarrow Z$ is étale,

$$\eta^* K_Z \sim K_{\bar{Z}}.$$

Thus $K_{\bar{Z}}$ is trivial. This implies $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is reduced by Lemma 4.3. \square

CLAIM 4.12. *If K_Z is trivial, the action of H/H' on \bar{Z} preserves two irreducible component or two intersections of irreducible components of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$. The morphism $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{D}$ is decomposed $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{A}$ and $\bar{A} \rightarrow \bar{D}$, where $\bar{A} \rightarrow \bar{D}$ is a smooth abelian fibration and every fibre of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{A}$ is a Kodaira singular fibre of type I_{2m} . All morphisms are H/H' -equivariant. Moreover there exists a smooth abelian fibration $\tilde{A} \rightarrow \bar{D}$ and an étale morphism $\tilde{A} \rightarrow \bar{A}$ over \bar{D} of degree two such that $(\bar{Z} \times_{\bar{\Delta}^n} \bar{D}) \times_{\bar{A}} \tilde{A}$ is isomorphic to the product of the Kodaira singular fibre of type I_{2m} and \tilde{A} .*

Proof. Let ϕ' be the isomorphism induced by the action of H/H' . By Claim 4.11, $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is reduced. Thus there exists a section o of $\bar{g} : \bar{Z} \rightarrow \bar{\Delta}$ and we fix o . Let $\bar{Z}' = \bar{Z} \setminus \bar{Z} \times_{\bar{\Delta}^n} \bar{D}$. Then \bar{Z}' is an abelian scheme and we define the automorphism ϕ on \bar{Z}' by $\phi' - \phi'(o)$. This automorphism defines a bimeromorphic map on \bar{Z} over $\bar{\Delta}^n$ by [11, Proposition 1.6]. Since \bar{Z} is the unique relative minimal model over $\bar{\Delta}^n$, it defines an automorphism on \bar{Z} , which is denoted by the same symbol. We will prove that ϕ has the following four properties:

- (1) ϕ has fixed points.
- (2) The induced automorphism on the dual graph of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$ is a reflection.

- (3) For every point $s \in \bar{D}$, the induced automorphism on the piece $H^{0,1}$ of the mixed Hodge structure of $H^1(\bar{Z}_s, \mathbb{C})$ is trivial, where \bar{Z}_s is the fibre at $s \in \bar{D}$.
- (4) ϕ^2 is trivial.

Before proving these properties, we remark that the assertions of Claim follows by the above properties. By the above four properties, ϕ defines an action of $\mathbb{Z}/2\mathbb{Z}$ on \bar{Z} which satisfies the assumptions of Claim 4.9. By Claim 4.9 (1), the number of irreducible components is even and this implies the first assertion of Claim. By Claim 4.9 (2), $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{D}$ is decomposed $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{A}$ and $\bar{A} \rightarrow \bar{D}$. Every fibre of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{A}$ is the Kodaira singular fibre of type I_{2m} and $\bar{A} \rightarrow \bar{D}$ is a smooth abelian fibration. These morphism is compatible with ϕ' , because every fibre of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{A}$ consists of \mathbb{P}^1 . This implies the second assertion of Claim. The last assertion of Claim follows by Claim 4.9 (3). Now we prove the properties (1), (2), (3) and (4). It is easy to see that ϕ has the property (1), because $\phi(o \cap \bar{Z}_s) = o \cap \bar{Z}_s$. To prove other properties, we fix notation. Let $\bar{\Delta} \subset \bar{\Delta}^n$ be a unit disk which is stable under the action of H/H' and intersects \bar{D} at s transversally. We define the morphism $\mathbb{H} \rightarrow \bar{\Delta}$ by $\tau \mapsto \exp(2\pi i\tau)$ and consider $\bar{Z}_\infty = (\bar{Z} \times_{\bar{\Delta}^n} \bar{\Delta}) \times_{\bar{\Delta}} \mathbb{H}$. Then $-\phi(o)$ induces an isomorphism \bar{Z}_∞ and the induced automorphism of $H^1(\bar{Z}_\infty, \mathbb{C})$ is trivial, because $-\phi(o)$ preserves every fibre of $\bar{Z}' \times_{\bar{\Delta}^n} \bar{\Delta} \rightarrow \bar{\Delta} \setminus \{s\}$ and defines a translation on it. Thus the induced automorphisms of ϕ^* and ϕ'^* on $H^1(\bar{Z}_\infty, \mathbb{C})$ coincide. Since $\bar{Z} \times_{\bar{\Delta}^n} \bar{\Delta} \rightarrow \bar{\Delta}$ is semistable, we have the Clemens-Schmid exact sequence

$$0 \rightarrow H^1(\bar{Z}_s, \mathbb{C}) \rightarrow H^1(\bar{Z}_\infty, \mathbb{C}),$$

which is ϕ' -equivariant and ϕ -equivariant. Thus the induced automorphisms of ϕ^* and ϕ'^* on $H^1(\bar{Z}_s, \mathbb{C})$ also coincide. This implies that ϕ has the properties (2) and (3). Since ϕ'^2 is trivial, the induced automorphism on $H^1(\bar{Z}_\infty, \mathbb{C})$ is trivial. Hence the induced automorphism $(\phi^2)^*$ on $H^1(\bar{Z}_\infty, \mathbb{C})$ is also trivial. Since ϕ^2 preserves the section o and every fibre of $\bar{Z} \times_{\bar{\Delta}^n} \bar{\Delta} \rightarrow \bar{\Delta}$, ϕ^2 is a homomorphism of the abelian scheme $\bar{Z}' \times_{\bar{\Delta}^n} \bar{\Delta} \rightarrow \bar{\Delta} \setminus \{s\}$. Therefore ϕ^2 is trivial on $\bar{Z}' \times_{\bar{\Delta}^n} \bar{\Delta}$. This implies ϕ^2 is trivial. \square

We go back to the proof of Lemma. We consider the case that H/H' preserves two irreducible components of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$. By Claim 4.12, $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{D}$ is decomposed $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{A}$ and $\bar{A} \rightarrow \bar{D}$ which are H/H' -equivariant. Since H/H' preserves every fibre $\bar{Z} \times_{\bar{\Delta}^n} \bar{D} \rightarrow \bar{D}$, it is enough to investigate the action on H/H' on \bar{A}_s and \bar{Z}_s which are fibres at $s \in \bar{D}$. Let $\bar{E}_{i,s}$, ($i = 1, 2$) be the preserved irreducible components of \bar{Z}_s . Since the action of H/H' on the piece $H^{0,1}$ of the mixed Hodge structure $H^1(\bar{Z}_s, \mathbb{C})$ is trivial, the action of H/H' on $H^1(\bar{E}_{i,s}, \mathbb{C})$, ($i = 1, 2$) is trivial by Lemma 4.2. Thus the induced action on \bar{A}_s from $\bar{E}_{i,s} \rightarrow \bar{A}_s$ is a nontrivial translation. Therefore the action of H/H' on $\bar{Z}_s \rightarrow \bar{A}_s$ is a translation on \bar{A}_s and a reflection on its fibre. This implies $Z \times_{\Delta^n} D$ satisfies all properties of Type I_m^*-2 in Table 1. The same argument shows that $Z \times_{\Delta^n} D$ satisfies all properties of Type I_m^*-3 in Table 1 if H/H' preserves two intersections of irreducible components of $\bar{Z} \times_{\bar{\Delta}^n} \bar{D}$. \square

We complete the proof of Proposition. \square

5. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. (1) Let $f : X \rightarrow \Delta^n$ be as in Theorem 1.3. By Proposition 2.7, we have a finite cyclic cover $\pi : \tilde{\Delta}^n \rightarrow \Delta^n$ with the Galois group H and a projective abelian fibration $\tilde{g} : \tilde{Z} \rightarrow \tilde{\Delta}^n$, which is smooth or of first order degenerations. Moreover \tilde{Z}/H is bimeromorphic to X . If \tilde{g} is smooth, then we have the unique relative minimal model Z of \tilde{Z}/H over Δ^n by Proposition 3.2. If \tilde{g} is of first order degenerations, we have the unique relative minimal model

Z of \tilde{Z}/H over Δ^n by Propositions 4.6 and 4.7. In both cases, Z is smooth. Since X is also a relative minimal model over Δ^n , $X \cong Z$. This implies that X is smooth.

(2) Since $X \cong Z$, $K_X = K_Z = \mathcal{O}$. Thus $X \times_{\Delta^n} D$ satisfies the properties one of types in Theorem 1.3 by Propositions 3.2, 4.6 and 4.7.

(3) We note that if we construct 4-dimensional examples then we obtain any dimensional examples by taking the product of a 4-dimensional example and a smooth Lagrangian fibration. In [10, §6], 4-dimensional examples of each types in Table 1 are given except of Type I_0^* -5 and I_m^* -3. First we construct an example of Type I_0^* -5. Let E be an elliptic curve which is isomorphic to $\mathbb{C}/\mathbb{Z} + i\mathbb{Z}$, H a finite cyclic group which is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and Δ a unit disk. We consider the quotient of $\tilde{Z} = E \times E \times \Delta \times \Delta$ by H , where the action of H is defined by

$$\begin{aligned} (x, y, u_1, u_2) &\xrightarrow{\alpha} \left(-ix, y + \frac{1}{4}, iu_1, u_2 \right) \\ &\xrightarrow{\beta} \left(x + \frac{1}{2} + \frac{i}{2}, y + \frac{1}{2}, u_1, u_2 \right). \end{aligned}$$

In the above expression, α is a generator of $\mathbb{Z}/4\mathbb{Z}$ and β is a generator of $\mathbb{Z}/2\mathbb{Z}$. Then the symplectic form $dx \wedge du_1 + dy \wedge du_2$ is H -equivariant and the singular locus of \tilde{Z}/H coincide with Type I_0^* -5 of Table 3. Thus we obtain a desired example by taking the minimal resolution of Du Val singularities of type A_1 . Next we construct an example of Type I_m^* -3. Let $R_k = \text{Spec}\mathbb{C}[u^{k+1}v^{-1}, u^{-k}v]$. Then R_k admits a fibration $R_k \rightarrow \text{Spec}\mathbb{C}[u]$. The affine varieties R_k and the morphisms $R_k \rightarrow \text{Spec}\mathbb{C}[u]$ are patched each together. We define

$$\tilde{P} := (\cup R_k) \times_{\text{Spec}\mathbb{C}[u]} \Delta$$

and the action of \mathbb{Z} on \tilde{P} by

$$u^a v^c \mapsto u^{a-mc} v^{-c}.$$

Then this action is properly discontinuous if the radius of Δ is sufficiently small. We denote by \tilde{Z} the quotient. Since the 2-form $1/vdu \wedge dv$ on \tilde{P} is equivariant under the action of \mathbb{Z} , \tilde{Z} is symplectic. Moreover \tilde{Z} admits a Lagrangian fibration over Δ because a general fibre of $\tilde{Z} \rightarrow \Delta$ is one dimensional. Let E be an elliptic curve. We consider the following two isomorphisms on the product $S \times E \times \Delta$.

$$\begin{aligned} (u^{k+1}v^{-1}, u^{-k}v, x, t) &\xrightarrow{\alpha} ((-1)^{k+1}u^{k+1}v, (-1)^{-k}u^{-k}v^{-1}, x, t) \\ (u^{k+1}v^{-1}, u^{-k}v, x, t) &\xrightarrow{\beta} \left(u^{k+2}v^{-1}, u^{-k-1}v, x + \frac{1}{4}, t \right) \end{aligned}$$

Then $\phi = \beta \circ \alpha$ defines an action of $\mathbb{Z}/4\mathbb{Z}$ on $S \times E \times \Delta$. This action has no fixed points. Moreover the symplectic form $1/vdu \wedge dv + dx \wedge dt$ is equivariant under this action. Thus the quotient of $(S \times E \times \Delta)/(\mathbb{Z}/4\mathbb{Z}) \rightarrow \Delta^2$ is a Lagrangian fibration. We show that this gives an example of Type I_m^* -3. Since ϕ^2 is expressed

$$(u^{k+1}v^{-1}, u^{-k}v, x, t) \mapsto \left(-u^{k+1}v^{-1}, -u^{-k}v, x + \frac{1}{2}, t \right),$$

the quotient $(\tilde{Z} \times E \times \Delta)/(\mathbb{Z}/2\mathbb{Z}) \rightarrow \Delta^2$ is of first order degenerations. Let D be the discriminant locus of $(\tilde{Z} \times E \times \Delta)/(\mathbb{Z}/2\mathbb{Z}) \rightarrow \Delta^2$ and Γ the dual graph of $(\tilde{Z} \times E \times \Delta)/(\mathbb{Z}/2\mathbb{Z}) \times_{\Delta^2} D$. We investigate the action of ϕ on Γ . The isomorphism α induces a reflection on Γ which preserves two vertices and β induces a rotation by which a vertices maps to the adjacent vertices. Thus ϕ induces a reflection which preserves two edges of Γ . This implies that $(S \times E \times \Delta)/(\mathbb{Z}/4\mathbb{Z}) \rightarrow \Delta^2$ gives an example of Type I_m^* -3. \square

Proof of Corollary 1.7. Since K_X is trivial, $f^*f_*K_{X/S} \sim -f^*K_S$. Thus we take $B = 0$ in [4, Proposition 2.2] and we have

$$K_X \sim f^*(K_S + L_{X/S}).$$

By [4, Definition 4.3],

$$L_{X/S} = L_{X/S}^{ss} + \sum (1 - t_P)P,$$

where P runs over codimension one points over S and

$$t_P := \max\{t \in \mathbb{R}; (X, tf^*P)$$

has only log canonical singularities in the generic point of $f^*P\}.$

Let P be an irreducible component of the discriminant locus D of f . It is easy to see that $(1 - t_P)$ coincide with a_P in Table 2 according to the type of f^*P . To prove that $12L_{X/S}^{ss}$ is Cartier, by [4, (3.3), (3.5) and (3.6)], it is enough to consider the base change $X \times_S \Delta$ and its semistable reduction, where $\Delta \subset S$ is a unit disk which intersects D transversally at a general point of D . Let $n = \dim S$. If we consider a polydisk $\Delta^n \subset S$ such that $\Delta \subset \Delta^n$ and $\Delta^n \cap D$ is smooth, the restriction $X \times_S \Delta^n$ has a semistable reduction by Proposition 2.7. We take the base change of the diagram in Proposition 2.7 by Δ and $\tilde{\Delta}$, where $\tilde{\Delta} = \pi^{-1}(\Delta)$. Let $\tilde{g}_{\tilde{\Delta}} : \tilde{Z} \times_{\tilde{\Delta}^n} \tilde{\Delta} \rightarrow \tilde{\Delta}$ be the morphism obtained by the base change and 0 its critical point of $\tilde{\Delta}$. The Galois group H acts on $(\tilde{g}_{\tilde{\Delta}})_* \mathcal{O}(K_{\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{\Delta}/\tilde{\Delta}}) \otimes \mathbb{C}(0)$ through a character $\chi : H \rightarrow \mathbb{C}^*$. By [4, (3.5)], $NL_{X/S}^{ss}$ is Cartier if $\chi^N = 1$. We consider $R^n(\tilde{g}_{\tilde{\Delta}})_* \mathcal{O}$ instead of $(\tilde{g}_{\tilde{\Delta}})_* \mathcal{O}(K_{\tilde{Z} \times_{\tilde{\Delta}^n} \tilde{\Delta}/\tilde{\Delta}})$ because H acts on $R^n(\tilde{g}_{\tilde{\Delta}})_* \mathcal{O} \otimes \mathbb{C}(0)$ through the complex conjugate $\bar{\chi}$ of χ . Since $\tilde{g}_{\tilde{\Delta}}$ is semistable, $R^n(\tilde{g}_{\tilde{\Delta}})_* \mathcal{O} \cong \wedge^n R^1(\tilde{g}_{\tilde{\Delta}})_* \mathcal{O}$. Thus we consider the action of H on $R^1(\tilde{g}_{\tilde{\Delta}})_* \mathcal{O} \otimes \mathbb{C}(0) \cong H^1(\tilde{Z}_0, \mathcal{O})$, where \tilde{Z}_0 is the fibre of $\tilde{g}_{\tilde{\Delta}}$ at 0. If $\tilde{g}_{\tilde{\Delta}}$ is smooth, $\dim H^1(\tilde{Z}_0, \mathcal{O})^H \geq n - 1$. By the argument in 3.7, the order of image of $H \rightarrow \text{Aut} H^1(\tilde{Z}_0, \mathcal{O})$ equals to 1, 2, 3, 4 or 6. Thus $\bar{\chi}^{12} = 1$. If $\tilde{g}_{\tilde{\Delta}}$ is of first order degenerations, the action of H on $H^{0,1}$ is trivial by Proposition 2.12. Let $\Gamma_{\tilde{Z}_0}$ be the dual graph of \tilde{Z}_0 . Then $H^{0,0} \cong H^1(\Gamma_{\tilde{Z}_0}, \mathbb{C})$. The action on the piece of $H^{0,0}$ is trivial if the action of H on $\Gamma_{\tilde{Z}_0}$ rotation. It is the multiplication by -1 if the action is a reflection. Thus $\bar{\chi}^2 = 1$ and we are done. \square

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